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A FIBER BUNDLE.*

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ABSTRACT

A model is developed for the failure time of a bundle of fibers subjected to a constant load. At any time, all surviving fibers share the bundle load equally while all failed fibers support no load. The bundle may collapse immediately or fibers may fail randomly in time, possibly more than one at a time. The failure time of the bundle is the failure time of the last surviving fiber. For a single fiber, the c.d.f. for the failure time is assumed to be a specific functional of an arbitrary load history. The model is developed using a quantile process approach. In the most important case the failure time of the bundle is shown to be asymptotically normal with known parameters. The bundle failure model has the features of both static strength and fatigue failure of earlier analyses, and thus is more realistic than earlier models.

FIBER BUNDLE; TIME TO FAILURE; LOAD SHARING; WEAK CONVERGENCE; QUANTILE PROCESSES

1. INTRODUCTION

Consider a bundle of n fibers to which we apply a nonnegative, time dependent load $\ell_s(t)$, $t \geq 0$. As time passes, fibers fail in a random manner that depends on their individual load histories. We let $T_1 \leq \ldots \leq T_n$ be the ordered failure times of the n fibers and designate T_n as the <u>bundle failure time</u>.

<u>Bundle loading assumptions</u>. The <u>bundle load program</u> is defined as $\ell(t) = \ell_S(t)/n$, $t \geq 0$, but the actual load that each fiber carries may differ from $\ell(t)$ because the load on a failed fiber is shifted to its survivors according to a specified rule. The <u>equal load sharing rule</u> is assumed in this study wherein all surviving fibers at time t share the load equally. Thus, we define the <u>actual fiber load program</u> L_n as the random process

$$L_n(t) = \begin{cases} \ell(t)/(1-i/n) & \text{for } T_i \leq t < T_{i+1} \text{ and } i=0,\ldots,n-1 , \\ 0 & \text{for } T_n \leq t , \end{cases} \tag{1.1}$$

where $T_0 \equiv 0$. Each fiber is subjected to $L_n(t)$ up to its time of failure. A large portion of this study will be devoted to studying bundle failure under the constant load program $\ell_1(t) = L$, $t \geq 0$ where L is a positive constant. Generally, we will assume $\ell(t)$, $t \geq 0$ to be continuous.

<u>Assumptions on single fiber failure</u>. We assume that fibers are sampled independently from a common source. Under the <u>known</u> arbitrary load program $\lambda(t)$, $t \geq 0$ a <u>single</u> fiber has random failure time T with cumulative distribution function (c.d.f.) $F(t|\lambda)$, $t \geq 0$ which is a nonanticipating functional of λ. Though some results will be obtained for general $F(t|\lambda)$, most of the study will involve the specific functional

$$F(t|\lambda) = \sup_{0\leq\tau\leq t} \{\Psi(\lambda(\tau), \textstyle\int_0^\tau \kappa(\lambda(s))ds)\} , \tag{1.2}$$

where $\kappa(x)$ and $\Psi(x,z)$ are functions with the following special properties.

The function $\kappa(x)$, $x \geq 0$ is assumed to be continuous, increasing and unbounded and to satisfy $\kappa(x) > 0$ for $x > 0$. We call $\kappa(x)$ the <u>breakdown rule</u>. An example of practical interest is the <u>power law breakdown rule</u>

$$\kappa_1(x) = x^\rho \quad \text{for} \quad x \geq 0, \tag{1.3}$$

where ρ is a positive constant. Later, we further restrict the behavior of $\kappa(x)$ in order to obtain asymptotic results for bundle failure.

The function $\Psi(x,z)$ is called the <u>shape function</u> and we assume it to be increasing and jointly continuous in $x \geq 0$ and $z \geq 0$, and to satisfy $\Psi(0,0) = 0$. Also, we assume $\Psi(x,z) \leq 1$ for $x \geq 0$ and $z \geq 0$ and $\lim_{x\to\infty} \Psi(x,z) = 1$ for all $z > 0$ <u>or</u> $\lim_{z\to\infty} \Psi(x,z) = 1$ for all $x > 0$. Later we further restrict the behavior of Ψ in order to obtain asymptotic results for bundle failure. An example of Ψ of practical interest is

$$\Psi_1(x,z) = 1-\exp\{-(x^r + z)^s\} \quad \text{for } x \geq 0 \text{ and } z \geq 0, \tag{1.4}$$

where r and s are positive constants.

In recent models for the fatigue failure of materials, c.d.f.'s for the failure time arise which are of the form of (1.2). In fact (1.2) with (1.3) and (1.4) is a specific example. In such models, fatigue cracks grow in length within the material and their growth depends on the load through the integral contained in (1.2). The strength of the material depends on the length of the longest crack and failure occurs when the load exceeds the strength <u>or</u> the strength decreases below the load for the first time. For periods where the load is decreasing, material strength may decrease less severely so that failure will not occur during that period. Thus, the "sup" in (1.2) arises as a result of these situations and is necessary to ensure that $F(t|\lambda)$ is increasing in t.

We will speak of the distribution for the initial strength of the fiber

(1.5) $$\bar{F}(x) = \Psi(x,0),\ x \geq 0\ .$$

Under the load program $\ell_1(t) = L,\ t \geq 0$ observe that $\bar{F}(L)$ is the probability that the fiber fails at time zero or alternatively that its initial strength is less than or equal to the constant L. When $\Psi = \Psi_1$, then $\bar{F}(x)$ is the Weibull distribution. Generally $\bar{F}(x),\ x \geq 0$ may not be a proper c.d.f. We will also speak of the distribution for the time to failure of the fiber in static fatigue

(1.6) $$F(t|\ell_1) = \Psi(L,\kappa(L)t) \text{ for } t \geq 0.$$

Note that $F(0|\ell_1) = \bar{F}(L)$ and that there may be a positive probability that the fiber fails immediately upon application of the load L. Under our assumptions, either $\bar{F}(x)$ or $F(t|\ell_1)$ is a proper c.d.f.

Our interest in this paper will be primarily in load programs which are increasing. In such cases we may drop the "sup" in (1.2) and take $\tau = t$. The integral in (1.2), which we call the fatigue integral, provides for the time dependent fatigue of the fiber, with κ influencing the rate as the load changes. It may be shown that (1.2) yields a probabilistic version of the well-known Miner's fatigue rule. A main feature of (1.2) is that $F(t|\lambda)$ is sensitive to the instantaneous value of the load λ. Consequently jumps in $\lambda(t)$ may result in corresponding jumps in F. Note that the function Ψ governs the shape of the distribution for initial strength $\bar{F}(x)$ and of the distribution for time to failure in static fatigue $F(t|L_1)$. Also Ψ governs the interactive influence on F of the instantaneous value of the load and of the fatigue integral. If $\Psi(x,z)$ is a function only of x, the fibers are called classic fibers and do not fatigue with time. Their strength distribution is $\bar{F}(x)$ which of course is time independent. For example, classic fibers are obtained that have a Weibull distribution for strength when y is deleted in (1.4).

<u>Previous bundle analysis</u>. The model studied earlier by Phoenix (1978) (under assumptions which were proposed by Coleman (1958)) is a special case of the present model. Specifically, the shape function $\Psi(x,z)$ was considered to be only a function of z. Consequently $F(t|\lambda)$ remained continuous in t under step increases in $\lambda(t)$ (although the slope of F at such times often increased drastically). Consequently fibers in the loaded bundle failed <u>one-at-a-time</u>, though generally at an accelerating rate. But more important, the model was incompatible with the classic (static) model for fiber bundle strength investigated in depth by Daniels (1945) for which $\Psi(x,z)$ is only a function of x. For a classic bundle, a linearly increasing load program $\ell(t)$ may result in more than one fiber failure at some time. This occurs because a shift of the load of a failed fiber to its survivors may cause some of the survivors to fail also, as their strengths are exceeded. At some time the classic bundle will collapse as the remaining survivors fail simultaneously.

Our bundle failure model has the features of the Daniels classic model and the Coleman time dependent model. A typical evolution of the bundle failure process under the constant load program $\ell_1(t)$, $t \geq 0$ will be that some fraction (perhaps all) of the fibers will fail at time zero upon application of the load (Daniels bundle feature). As time passes single fibers will fail (Coleman feature) perhaps triggering instantaneously one or more additional failures as a result of the load jumps (Daniels feature). The result is that ties will occur among the fiber failure times. At some time, the remaining fibers will fail simultaneously as the bundle collapses catastrophically (Daniels feature).

In this paper we formulate a model of the bundle failure process whose statistical properties will be equivalent to those of the bundle as originally described. This model will be called the <u>quantile model</u> because the time to failure of the bundle T_n will be expressed in terms of the <u>uniform quantile process</u>. In fact, for the constant bundle load program $\ell_1(t)$ we will obtain T_i

for $i = 1, \ldots, n$ as a functional of the uniform quantile process. Under broad assumptions we will demonstrate asymptotic normality for the bundle failure time T_n using techniques of analysis adapted from Shorack (1972a and b). An example of engineering relevance will be discussed to highlight the main differences between the model of Phoenix (1977) and the more general model considered here. But first we consider on a heuristic basis some examples of infinite ($n \to \infty$) bundles. These will illustrate several of the features of the bundle failure process and will provide motivation for the exact and asymptotic analyses which follow.

2. INFINITE BUNDLE ANALYSIS

Consider an infinitely large bundle ($n\to\infty$), which is subjected to the constant load program $\ell_1(t) = L$, $t \geq 0$. (The actual load on the bundle is $\ell_S(t) = nL$ which is, of course, infinitely large). Upon application of the load L, at time zero, the fraction $y_1 = \bar{F}(L)$ immediately fails being unable to support L. (Recall that $\bar{F}(x)$ given by (1.5) is the initial strength distribution of the fiber.) Immediately, the load on each surviving fiber jumps to $L/(1-y_1)$ and the fraction of failed fibers increases instantaneously to $y_2 = \bar{F}(L/(1-y_1))$. This process continues at time zero and after the m^{th} round, the fraction of failed fibers is $y_m = \bar{F}(L/(1-y_{m-1}))$. In Figure 1 we give an example and illustrate four possibilities associated with the load levels L_1, L_2, L_3 and L^* respectively. For $L = L_1$ we depict the generation of the sequence $y_1, y_2, \ldots$ and note that $y_m \to y_1^{\#} < 1$ as $m \to \infty$ so that $y_1^{\#}$ becomes the stable fraction of fibers that fail at *time zero*. On the other hand, if $L = L_2$ we have $y_m \to 1$ as $m \to \infty$ so that <u>all</u> the fibers fail at time zero. But for $L = L_3$ <u>no</u> fibers fail and $y_m = 0$ for all m. The critical value of L is the load L^* for which the curve $\bar{F}(L^*/(1-y))$ touches (at $y = y^*$) but does not cross the curve y. Thus for loads exceeding L^* the bundle collapses. Now Daniels (1945) demonstrated that under reasonable assumptions on $\bar{F}(x)$ the strength of a classic bundle is asymptotically normally distributed with mean $\mu_{max} = \sup\{x(1-\bar{F}(x));\ x \geq 0\}$ and variance which decreases as $1/n$ (see Phoenix and Taylor (1973)). Taking x^* as the point (assumed unique) where the function $x(1-\bar{F}(x))$, $x \geq 0$ achieves its maximum we find indeed that $L^* = \mu_{max}$ and $x^* = L^*/(1-y^*)$. Also the fact that the variance decreases as n^{-1} supports our heuristic approach. Thus, we have the following situation at time zero. We have $L^* > 0$ as the smallest value of L for which $\bar{F}(L/(1-y)) \geq y$ for all y, if this value exists or take $L^* = \infty$ otherwise. We call L^* the <u>initial collapse load</u> for the infinite bundle. We have $y_1^{\#} \in [0,1]$ as the smallest value of $y \geq 0$ for which $\bar{F}(L/(1-y)) = y$ and call $y_1^{\#}$ the <u>initial fraction of failed</u>

<u>fibers in the infinite bundle</u>. As an example if $\Psi = \Psi_1$ we find that $\bar{F}(x)$ is the two parameter Weibull c.d.f. with shape parameter rs. We obtain

$$(2.1) \qquad L^* = (rs)^{-1/(rs)} \exp\{-1/(rs)\}$$

and $y_1^\#$ as the smallest value of $y > 0$ satisfying

$$(2.2) \qquad \log\left(\frac{1}{1-y}\right) = \left(\frac{L}{1-y}\right)^{rs}$$

when $L \leq L^*$, and $y_1^\# = 1$ otherwise.

Let $y(t)$, $t \geq 0$ be the fraction of failed fibers at time t. Since failed fibers cannot repair themselves, $y(t)$ will be increasing in $t \geq 0$ and will satisfy $y(0) = y_1^\#$. If $y(t)$ becomes unity for finite t, we define the <u>bundle failure time</u> $t_\infty(1)$ (for infinite bundles) as

$$(2.3) \qquad t_\infty(1) = \inf\{t \geq 0;\ y(t) = 1\} .$$

Otherwise $t_\infty(1)$ is taken as infinity. As time passes, equilibrium is maintained and we argue that $y(t)$, $t \geq 0$ must satisfy the integral equation

$$(2.4) \qquad y(t) = \Psi(L/(1-y(t)),\ \int_0^t \kappa(L/(1-y(\tau))\, d\tau)$$

where for all $t \geq 0$, $y(t)$ is the smallest possible value. Note that each fiber of the bundle is subjected to $L/(1-y(t))$ up to its time of failure and by (1.2) the right hand side of (2.4) will be the fraction of fibers that has failed up to time t. For $t_\infty(1)$ to be finite when $0 < L \leq L^*$ we require $\Psi(L,z) \to 1$ as $z \to \infty$ and Ψ and κ together will be required to have special properties.

Given $0 \le y < 1$ and $x > 0$ let $\Psi^{-1}(x,y)$ be the largest value of z satisfying $\Psi(x,z) = y$, if such a value exists and let $\Psi^{-1}(x,y)$ equal zero otherwise. Note that $\Psi(x,z)$ may be viewed as a c.d.f. in $z \ge 0$ given $x > 0$ and that $\Psi^{-1}(x,y)$ is its inverse in y. Observe also that $\Psi^{-1}(x,y)$ is increasing in $0 < y < 1$, is decreasing in $x > 0$, and is jointly continuous in both variables. For example when $\Psi = \Psi_1$ we have

$$(2.5) \qquad \Psi_1^{-1}(x,y) = \begin{cases} [\log(\frac{1}{1-y})]^{1/s} - x^r & \text{for } 0 \le y < 1 \text{ and } 0 \le x \le [\log(\frac{1}{1-y})]^{\frac{1}{rs}} \\ 0 \,, & \text{otherwise.} \end{cases}$$

Proceeding with the solution to (2.4) for $0 < L \le L^*$ we rearrange (2.4) to obtain

$$(2.6) \qquad \int_0^t \kappa\left(\frac{L}{1-y(\tau)}\right)d\tau = g(y(t))$$

where

$$(2.7) \qquad g(y) = \Psi^{-1}(L/(1-y),y) \quad \text{for} \quad 0 \le y < 1 \,,$$

and (2.6) holds for some range of t. Now the left hand side of (2.6) is strictly increasing in $t \ge 0$ by the nonnegativity assumption of κ, but the behavior of $g(y)$ is more complex. Although $\Psi^{-1}(x,y)$ is increasing in y, it is <u>decreasing</u> in x. But we also see that $g(y) > 0$ for $0 < y < 1$ if and only if $\bar{F}(L/(1-y)) < y$. Thus for $0 \le y < y_1^{\#}$ we must have $g(y) = 0$. For $y \ge y_1^{\#}$ we find that $g(y)$ is not necessarily increasing. We restrict the behavior of $g(y)$ as follows: Assume that given $0 < L \le L^*$ there exist points $0 \le y_1^{\#} \le y_2^{\#} \le y_3^{\#} \le 1$ such that $g(y)$ is is zero on $[0,y_1^{\#})$, is strictly increasing on $[y_1^{\#}, y_2^{\#})$, is strictly decreasing on

$[y_2^{\#}, y_3^{\#})$ and is zero on $[y_3^{\#}, 1)$. Possibly $y_1^{\#} = 0$ and possibly $y_3^{\#} = 1$. Also $y_2^{\#}$ may be zero or one as well. These restrictions on the behavior of $g(y)$ turn out to be mild from a practical point of view. Returning to (2.6) we may write

$$\text{(2.8)} \qquad dt = \phi(y(t))dg(y(t)) \quad \text{for} \quad y_1^{\#} \leq y < y_2^{\#}$$

where

$$\text{(2.9)} \qquad \phi(y) = 1/\kappa(L/(1-y)) \quad \text{for} \quad 0 \leq y < 1 \, ,$$

and $t = g(y_1^{\#})\phi(y_1^{\#})$ at $y = y_1^{\#}$. (When $y_1^{\#} = 0$ we may have $g(y_1^{\#}) > 0$ but for $y_1^{\#} > 0$ we have $g(y_1^{\#}) = 0$.) Henceforth we restrict ϕ to being right continuous on $[0,1)$. Denoting $t_\infty(y)$ as the solution, we integrate (2.8) to obtain

$$\text{(2.10)} \qquad t_\infty(y) = \int_{y_1^{\#}}^{y} \phi(z)dg(z) + g(y_1^{\#})\phi(y_1^{\#}) \quad \text{for} \quad y_1^{\#} \leq y < y_2^{\#} \, ,$$

which is a relationship between time t and the fraction of failed fibers y up to $y_2^{\#}$. The significance of $y_2^{\#} < 1$ is that when $y(t)$ reaches $y_2^{\#}$, the bundle collapses instantaneously as $y(t)$ jumps to one. We call $y_2^{\#}$ the <u>collapse fraction</u>. This is seen from the fact that the right hand side of (2.6) can grow no further while the left hand side continues to grow. Indeed (2.6) is no longer applicable and $y(t) = 1$ is the only solution to (2.4) beyond this time. Now for $L = L^*$ we note that $g(y) = 0$ for $0 \leq y < 1$. Thus the fraction of failed fibers $y(t)$ becomes y^* at $t = 0$ and is one for $t > 0$. Hence,

$$\text{(2.11)} \qquad t_\infty(1) = \begin{cases} \int_{y_1^{\#}}^{y_2^{\#}} \phi(y)dg(y) + \phi(y_1^{\#})g(y_1^{\#}) & \text{for} \quad 0 < L < L^* \, , \\ \\ 0 & \text{for} \quad L^* \leq L \, . \end{cases}$$

Letting $g^{\#}(y) = \sup\{g(\tau);\ 0 \leq \tau \leq y\}$ we integrate by parts in (2.10) and obtain by the above discussion the time to failure of the fraction y

$$(2.12) \qquad t_\infty(y) = -\int_0^1 g^{\#}(\min(z,y))d\phi(z) \quad \text{for} \quad 0 < L \quad \text{and} \quad 0 < y \leq 1 \ ,$$

where we use the fact that $\phi(y) \to 0$ as $y \to 1$ (by our assumptions on κ). Note that $t_\infty(y) = t_\infty(1)$ for $y_2^{\#} \leq y \leq 1$. Before discussing an example, we point out that $t_\infty(1)$ will be found to be the <u>mean of the asymptotic distribution of the bundle failure time</u> T_n when n grows large. Additional restrictions that we impose on Ψ^{-1}, ϕ and g will be mild. We also point out that $t_\infty(1)$ is finite if g(y) is bounded since $\phi(y)$ is always bounded. On the other hand if g(y) grows unbounded in y (so that $y_2^{\#} = 1$) we see from (2.11) that $t_\infty(1)$ may not exist. Later, reasonable restrictions on ϕ and g will foreclose this possibility. Note as well that in the model of Phoenix (1978) $\Psi(x,z) = \bar{\Psi}(z)$, a function only of z. Thus, $g(y) = \bar{\Psi}^{-1}(y)$ the inverse of $\bar{\Psi}$, the initial failed fraction $y_1^{\#}$ is zero, the collapse fraction $y_2^{\#}$ is one, and $L^* = \infty$.

In Figure 2 we have illustrated some features of $\Psi_1^{-1}(x,y)$ given by (2.5). Above the line $x = [\log(1/(1-y))]^{1/(rs)}$, which is the inverse $x = \bar{F}^{-1}(y)$ of the initial strength distribution $\bar{F}(x)$, we see that $\Psi_1^{-1}(x,y)$ is zero. Also we see that $\Psi_1^{-1}(x,y)$ is jointly continuous in both $x > 0$ and $0 \leq y < 1$. Now g(y) is $\Psi_1^{-1}(x,y)$ evaluated along the line $x = L/(1-y)$. We compute for $0 < L < L^*$ that

$$(2.13) \qquad g(y) = \begin{cases} 0 \text{ for } 0 \leq y < y_1^{\#} \text{ and } y_3^{\#} \leq y < 1, \\ \\ [\log(\frac{1}{1-y})]^{1/s} - (\frac{L}{1-y})^r \text{ for } y_1^{\#} \leq y < y_3^{\#} \ , \end{cases}$$

where L^* was given before by (2.1) and $y_1^\#$ and $y_3^\#$ are the smallest and largest solutions respectively to (2.2). The value L^* and the associated tangent point y^* are easily visualized and for $L > L^*$ we see $g(y) = 0$ for $0 \leq y < 1$. For $0 < L < L^*$ the function $g^\#(y)$ is

$$(2.14) \qquad g^\#(y) = \begin{cases} g(y) & \text{for } 0 \leq y \leq y_2^\# \\ g(y_2^\#) & \text{for } y_2^\# \leq y \leq 1 \end{cases}$$

where $y_2^\#$ is the positive solution to

$$(2.15) \qquad [\log(\frac{1}{1-y})]^{\frac{1-s}{s}} = rs(\frac{L}{1-y})^r .$$

Also $g^\#(y) = 0$ for $L \geq L^*$. Evidently for $0 < L < L^*$ we have $0 < y_1^\# < y_2^\# < y_3^\# < 1$, and the behavior of $g(y)$ is typical. Thus, we find that if $L \geq L^*$ the bundle fails immediately. But if $0 < L < L^*$, a positive fraction $y_1^\#$ of fibers fails at time zero. As time passes, fibers fail smoothly until the fraction $y_2^\#$ has failed when the bundle suddenly collapses. For this example, if we assume the <u>power law breakdown rule</u> $\kappa_1(x) = x^\rho$, we find that

$$(2.16) \qquad \phi(y) = L^{-\rho}(1-y)^\rho \quad \text{for} \quad 0 \leq y < 1.$$

Clearly $t_\infty(1)$ as computed by (2.11) or (2.12) is finite for all $\rho > 0$. But if the <u>direct load sensitivity feature is removed</u> as in Phoenix (1978) and Ψ_1 is replaced by $1-\exp\{-z^s\}$ for $z \geq 0$, we find $g(y) = g^\#(y) = \log(\frac{1}{1-y})^{1/s}$ for $0 \leq y < 1$ and essentially the first term in (2.13) is retained with $y_1^\# = 0$ and $y_2^\# = y_3^\# = 1$. Again $t_\infty(1)$ is finite for all $\rho > 0$ though the asymptotic distribution results there required $\rho \geq 1$. Evidently the load sensitivity feature will reduce the magnitude of $t_\infty(1)$ substantially when L nears L^*.

In the foregoing heuristic analysis we have introduced several quantities, and established some important features of bundle failure through the use of an example. Later we investigate the asymptotic distribution of T_n, the bundle failure time, and these as well as other quantities will arise. Further assumptions will be made along the lines of those made heretofore. These assumptions are rather innocuous in applications yet the proofs are kept as straightforward as is practicable. Certainly (2.12) holds for g for which there are more than two intersections of the functions $\bar{F}(L/(1-y))$ and y on $[0,1)$. However, such situations are rare in applications and only serve to complicate the proofs.

3. THE QUANTILE MODEL OF BUNDLE FAILURE.

We formulate a model of bundle failure whose statistical properties will be equivalent to those of the bundle as originally described. With the n fibers we associate the ordered sample $V_1 \leq \dots \leq V_n$ from the uniform distribution on [0,1]. Associated with the sample is the random process

$$(3.1) \qquad \Gamma_n^{-1}(y) = \begin{cases} V_i & \text{for } (i-1)/n \leq y < i/n \text{ and } i=1,\dots,n, \\ V_n & \text{for } y = 1, \end{cases}$$

and the uniform quantile process

$$(3.2) \qquad V_n(y) = \sqrt{n}\,(\Gamma_n^{-1}(y)-y) \quad \text{for} \quad 0 \leq y \leq 1.$$

We assume the bundle load program $\ell(t)$, $t \geq 0$ to be nonnegative and continuous. We recall that $F(t|\lambda)$, $t \geq 0$ was the c.d.f. of the time to failure for a single fiber given an arbitrary load history $\lambda(t)$, $t \geq 0$. We also recall that $F(t|\lambda)$ was a nonanticipating functional of λ.

Let $L_{n,0}(s) = \ell(s)$ for $s \geq 0$ and let

$$(3.3) \qquad L_{n,i}(s) = \begin{cases} \ell(s)/(1-(j-1)/n) & \text{for } T'_{j-1} \leq s < T'_j \text{ and } j=1,\dots,i \\ \ell(s)/(1-i/n) & \text{for } T'_i \leq s. \end{cases}$$

Now set $T'_0 \equiv 0$ and generate the quantile model failure times $T'_1 \leq \dots \leq T'_n$ by

$$(3.4) \qquad T'_i = \inf\{t \geq T'_{i-1};\ F(t|L_{n,i-1}) \geq V_i\}$$

when such a value exists and set $T_i' = \infty$ otherwise. Thus, we have the failure times $T_1' \leq \ldots \leq T_n'$ in terms of the uniform order statistics $V_1 \leq \ldots \leq V_n$ and later we will see that <u>the random vector</u> $(T_1',\ldots,T_n')$ <u>and the random vector</u> $(T_1,\ldots,T_n)$ <u>have the same distribution</u>.

When ℓ is the constant load program $\ell_1(t) = L$, $t \geq 0$ and $F(t|\lambda)$ is given by (1.2), we may express $T_1',\ldots,T_n'$ <u>explicitly</u> in terms of $V_1,\ldots,V_n$. This forms the basis for studying the asymptotic behavior of the bundle failure time for this important case. We have $T_0' = 0$,

$$(3.5) \qquad T_i' = \inf\{t \geq T_{i-1}';\ \Psi(L_{n,i-1}(t),\ \textstyle\int_0^t \kappa(L_{n,i-1}(s))ds) \geq V_i\}$$

where L replaces $\ell(s)$ in (3.3) and $L_{n,0} = L$. By our assumptions $T_1' \leq \ldots \leq T_n'$ are finite when $V_1 \leq \ldots \leq V_n < 1$. We restrict our attention to the fatigue case where $\Psi(x,z) \to 1$ as $z \to \infty$, and assume for technical simplicity that $\Psi(x,z)$ is <u>strictly</u> increasing in $z \geq 0$. Thus given $0 \leq y < 1$ and $x > 0$ we now have $\Psi^{-1}(x,y)$ as the value $z \geq 0$ satisfying $\Psi(x,z) = y$ when such a value exists, and $\Psi^{-1}(x,y) = 0$ otherwise. Thus, Ψ^{-1} is nonnegative. Now let

$$(3.6) \qquad W_i = \Psi^{-1}(nL/(n-i+1),\ V_i) \text{ for } i=1,\ldots,n$$

and associate with the W_i's the random process

$$(3.7) \qquad W_n(y) = \begin{cases} W_i \text{ for } (i-1)/n \leq y < i/n \text{ and } i=1,\ldots,n, \\ \\ W_n \text{ for } y = 1. \end{cases}$$

Note that $W_n(y)$ is nonnegative on $[0,1]$. Let

$$\gamma_n(y) = \begin{cases} (i-1)/n \text{ for } (i-1)/n \le y < i/n \text{ and } i=1,\ldots,n \\ \\ (n-1)/n \text{ for } y = 1 \end{cases} \tag{3.8}$$

and observe that

$$W_n(y) = \Psi^{-1}(L/(1-\gamma_n(y)),\ \Gamma_n^{-1}(y)) \text{ for } 0 \le y \le 1. \tag{3.9}$$

Of major importance is the random process

$$W_n^{\#}(y) = \sup\{W_n(\tau);\ 0 \le \tau \le y\} \text{ for } \ 0 \le y \le 1. \tag{3.10}$$

Note that $W_n^{\#}(y)$ may differ from $W_n(y)$ since the latter is not necessarily increasing in y. We may express the failure times $T_1',\ldots,T_n'$ in terms of $W_n^{\#}$. Specifically, we may write (3.5) as

$$\begin{aligned} T_i' = \inf\{t \ge T_{i-1}';\ &\kappa(L)T_1' + \kappa(nL/(n-1))(T_2' - T_1') \\ &+\ldots+ \kappa(nL/(n-i+1))(t-T_{i-1}') \ge W_i\} \end{aligned} \tag{3.11}$$

for $i=1,\ldots,n$. Inspecting (3.11) we see that $T_i' - T_{i-1}' > 0$ if and only if $W_i > \max(W_j;\ j=0,\ldots,i-1)$ where $W_0 \equiv 0$ and $i = 1,\ldots,n$. Hence we may write $\kappa(L)T_1' = W_n^{\#}(0)$ and

$$\kappa(nL/(n-i+1))(T_i' - T_{i-1}') = W_n^{\#}((i-1)/n) - W_n^{\#}((i-2)/n) \tag{3.12}$$

for i=1,...,n. Now recall the function $\phi(y) = 1/\kappa(L/(1-y))$ on $[0,1)$ where $L > 0$, and assume for technical simplicity that its derivative ϕ' exists on $(0,1)$. Thus, we combine (3.12) for i=1,...,n to yield

$$T_i' = -\int_0^{(i-1)/n}\phi'(z)W_n^{\#}(z)dz + \phi((i-1)/n)W_n^{\#}((i-1)/n). \tag{3.13}$$

Hence, we introduce

$$T_n'(y) = -\int_0^1 \phi'(s)W_n^{\#}(s\wedge y)ds. \tag{3.14}$$

where $s\wedge t \equiv \min(s,t)$. Then

$$T_i' = T_n'(y) \text{ for } (i-1)/n \leq y < i/n \text{ and } i=1,\dots,n, \tag{3.15}$$

since $\phi(y) \to 0$ as $y \to 1$. Also, we see that $T_n' = T_n'(1)$ so that $T_n'(1)$ will be the bundle failure time . Equation (3.14) serves as the starting point for determining the asymptotic distribution of the bundle failure time and of the time to failure of a given fraction y of fibers in the bundle which will be $T_n'(y)$.

Associated with the processes W_n and $W_n^{\#}$ are the functions

$$g_n(y) = \Psi^{-1}(L/(1-\gamma_n(y)),y) \text{ for } 0 \leq y < 1 \tag{3.16}$$

and

$$g_n^{\#}(y) = \sup\{g_n(\tau);\ 0 \leq \tau \leq y\} \text{ for } 0 \leq y < 1, \tag{3.17}$$

both of which are nonnegative. Two normalized processes play a central role. These are

$$Z_n(y) = \sqrt{n}\,\{W_n(y) - g_n(y)\} \text{ for } 0 \leq y < 1 \tag{3.18}$$

and

$$Z_n^{\#}(y) = \sqrt{n}\,\{W_n^{\#}(y) - g_n^{\#}(y)\} \text{ for } 0 \le y < 1 . \tag{3.19}$$

Most important however, are

$$t_n(y) = -\int_0^1 \phi'(s)g_n^{\#}(s \wedge y)ds \text{ for } 0 < y \le 1 \tag{3.20}$$

and $\mathcal{T}_n'(y) \equiv \sqrt{n}\,\{T_n'(y) - t_n(y)\}$. By (3.14), (3.17), (3.19) and (3.20) we have

$$\mathcal{T}_n'(y) = -\int_0^1 \phi'(s)Z_n^{\#}(s \wedge y)ds \quad \text{for} \quad 0 < y \le 1. \tag{3.21}$$

The normalized bundle failure time $\mathcal{T}_n'$ is defined as

$$\mathcal{T}_n' = \sqrt{n}\,\{T_n' - t_n(1)\} . \tag{3.22}$$

By (3.14), (3.19), (3.20) and (3.21) we have $\mathcal{T}_n' = \mathcal{T}_n'(1)$. The time to failure of the yth fraction of fibers is defined as $T_n'(y)$ and in normalized fashion by $\mathcal{T}_n'(y)$.

Later we will determine the asymptotic distributions for the normalized bundle failure time $\mathcal{T}_n'(1)$ and for $\mathcal{T}_n'(y)$ the normalized time to failure of the yth fraction of fibers. We will find three cases of interest depending on the value of the applied load L relative to a critical collapse value. For the most important case $\mathcal{T}_n'(1)$ and $\mathcal{T}_n'(y)$ will be asymptotically normal. In the next section we outline these cases and introduce several functions and random processes which are important in the development. We also introduce the random variables to which $\mathcal{T}_n'(1)$ and $\mathcal{T}_n'(y)$ will converge in distribution.

4. LIMITING RANDOM VARIABLES AND IMPORTANT FUNCTIONS

Earlier we introduced the functions Ψ and Ψ^{-1}. Henceforth we view $\Psi^{-1}(x,y)$ as a nonnegative and continuous function on the set S which is

$$S = \{(x,y);\ x>0 \text{ and } 0 \leq y<1\}\ . \tag{4.1}$$

We recall $\bar{F}(x) = \Psi(x,0)$ for $x \geq 0$ as the c.d.f. for initial fiber strength. By our assumptions $\bar{F}(x)$ is continuous and increasing with $\bar{F}(0) = 0$, but $\bar{F}$ may not be a proper c.d.f. ($\bar{F}(x) = 0$ for all $x \geq 0$ is possible). Now $y = \bar{F}(x)$ defines a very important line in S since it divides S into

$$S^0 = \{(x,y);\ x > 0 \text{ and } 0 < y < \bar{F}(x)\}\ , \tag{4.2}$$

and

$$S^+ = \{(x,y);\ x > 0 \text{ and } \bar{F}(x) < y < 1\}\ . \tag{4.3}$$

We let

$$\bar{S} = S^0 \cup S^+ \tag{4.4}$$

and see that $\bar{S}$ is just S with the lines $y = 0$ and $y = \bar{F}(x)$ removed. We find that Ψ^{-1} is <u>zero</u> on S^0 and is <u>positive</u> and <u>strictly increasing</u> in y on S^+.

In the infinite bundle analysis we introduced the function

$$g(y) = \Psi^{-1}(L/(1-y),y) \quad \text{for } 0 \leq y < 1. \tag{4.5}$$

The behavior of g is strongly influenced by the path in S that the line $x = L/(1-y)$ takes relative to the line $y = \bar{F}(x)$. Three cases of interest are possible depending on the value of the load L. Recall L^* as the load L for which the line $y = \bar{F}(x)$ touches from below, (but does not cross) the line $x = L/(1-y)$ at a single point in S. When such a value does not exist, we take $L^* = \infty$. We called L^* the initial collapse load.

Case I. (Applied load below initial collapse load). The line $y = \bar{F}(x)$ crosses the line $x = L/(1-y)$ once or twice in S at distinct points, and elsewhere the two lines do not touch. Thus $L < L^*$. We let $y_1^{\#}$ be the value of y at the first crossing and let $y_3^{\#}$ be the value of y at the second crossing if the second crossing occurs and let $y_3^{\#} = 1$ otherwise. Note that $y_1^{\#} < y_3^{\#}$, and only for y in $(y_1^{\#}, y_3^{\#})$ does the line $x = L/(1-y)$ lie in S^+. Hence we have $g(y) > 0$ only on $(y_1^{\#}, y_3^{\#})$. We assume that a point $y_2^{\#}$ exists such that $y_1^{\#} < y_2^{\#} \leq y_3^{\#}$ and $g(y)$ is strictly increasing on $[y_1^{\#}, y_2^{\#})$ and is strictly decreasing on $[y_2^{\#}, y_3^{\#})$. Thus for the points $0 \leq y_1^{\#} < y_2^{\#} \leq y_3^{\#} \leq 1$ we have that $g(y)$ is nonnegative and continuous on $[0,1)$, is zero on $[0,y_1^{\#})$ and on $[y_3^{\#},1)$, is strictly increasing on $[y_1^{\#}, y_2^{\#})$ and is strictly decreasing on $[y_2^{\#}, y_3^{\#})$.

Case II. (Applied load equals critical collapse load). The line $y = \bar{F}(x)$ touches the line $x = L/(1-y)$ at a single point in S. We let $y_2^{\#}$ be the point y of touching and we assume $0 < y_2^{\#} < 1$. For this case we naturally take $y_1^{\#} = y_2^{\#} = y_3^{\#}$ and have L^* as the value of L. Note that $g(y) = 0$ for $[0,1)$.

Case III. (Applied load exceeds critical collapse load). The lines $y = \bar{F}(x)$ and $x = L/(1-y)$ do not cross (or touch) in S. Thus $g(y) = 0$ on $[0,1)$ and we set $y_1^{\#} = y_2^{\#} = y_3^{\#} = 1$. We must have $L > L^*$.

Given Ψ, if $L^* = \infty$, we have Case I for all $L > 0$. For some Ψ, it is possible for more than two crossings of $y = \bar{F}(x)$ and $x = L/(1-y)$ to occur. For simplicity we ignore cases arising from this situation though extension of the results to these is straightforward. We also ignore Case II but with $y_2^{\#} = 0$.

Cases I to III cover nearly all situations of practical importance.

Some important functions. Associated with g are the functions

(4.6) $$g^{\#}(y) = \sup\{g(\tau);\ 0 \leq \tau \leq y\} \text{ for } 0 \leq y < 1$$

and

(4.7) $$y^{\#}(y) = y \wedge y_2 \quad \text{for} \quad 0 \leq y < 1.$$

Note that $g^{\#}(y) = g(y^{\#}(y))$. Let R be the set

(4.8) $$R = (0, y_1^{\#}) \cup (y_1^{\#}, y_3^{\#}) \cup (y_3^{\#}, 1) .$$

Later we will require the existence of continuous partial derivatives of Ψ^{-1} on $\bar{S}$ and we will assume regularity properties for them. Here we introduce

(4.9) $$g^{\partial}(y) = \partial\Psi^{-1}(x,y)/\partial y \Big|_{x=L/(1-y)} \quad \text{for } y \text{ in } R ,$$

which is the partial derivative of Ψ^{-1} with respect to y evaluated along the line $x = L/(1-y)$ in $\bar{S}$. It is easily seen that $g^{\partial}(y)$ is zero on $(0, y_1^{\#})$ and $(y_3^{\#}, 1)$, and is continuous and nonnegative elsewhere on R.

In the infinite bundle analysis we introduced the function $\phi(y) = 1/\kappa(L/(1-y))$ on $[0,1)$ where $L > 0$. By the properties of κ stated earlier, ϕ is bounded, continuous and decreasing on $[0,1)$ and satisfies $\phi(y) > 0$ for $0 \leq y < 1$. Also $\phi(y) \to 0$ as $y \to 1$. Later we will require the existence of the derivative ϕ' and will assume certain regularity properties for ϕ'.

Some important Gaussian processes. Let $\{V(y);\ 0 \le y \le 1\}$ be the Brownian bridge with mean zero and covariance function $s \wedge t - st$. Related to V is the Gaussian process $\{Z(y);\ y \in R\}$ where

$$(4.10) \qquad Z(y) = g^{\partial}(y)V(y) \quad \text{for} \quad y \text{ in } R\ .$$

The covariance function for Z is

$$(4.11) \qquad \Gamma(s,t) = (s \wedge t - st)g^{\partial}(s)g^{\partial}(t) \text{ for } s \text{ and } t \text{ in } R,$$

and the mean for Z is zero. Also arising is the Gaussian process $\{Z(y^{\#}(y));\ y \in R\}$ with mean zero and covariance function $\Gamma(y^{\#}(s),\ y^{\#}(t))$.

The random variables $T(y)$. For Case I ($L < L^{*}$) and $0 < y \le 1$ let

$$(4.12) \quad T_{I}(y) = \begin{cases} -\int_0^1 \phi'(s)Z(y^{\#}(s \wedge y))ds \text{ for } 0 < y \le 1 \text{ except at } y = y_1^{\#}\ , \\ \\ \phi(y_1^{\#})g^{\partial}(y_1^{\#})^{+}\max(V(y_1^{\#}),\ 0) \text{ for } y = y_1^{\#}\ , \end{cases}$$

where

$$(4.13) \qquad g^{\partial}(y)^{+} = \lim_{z \to y} \partial\Psi(L/(1-y),z)/\partial z \quad \text{and} \quad 0 < y < z < 1.$$

Also for $0 < y \le 1$ let

$$(4.14) \qquad \sigma^2(y) = \int_0^1 \int_0^1 \phi'(s_1)\phi'(s_2)\Gamma(y^{\#}(s_1 \wedge y),\ y^{\#}(s_2 \wedge y))ds_1 ds_2\ .$$

Note that $T_I(y)$ and $\sigma^2(y)$ are zero for y in $(0,y_1^{\#})$. Also $T_I(y)$ is a normal random variable for y in $(y_1^{\#}, 1]$ with mean zero and variance $\sigma^2(y)$ that will be finite by our assumptions. But for $y = y_1^{\#}$, we have $T_I(y_1^{\#}) \geq 0$ and $P\{T_I(y_1^{\#}) = 0\} = 1/2$. Also, $P\{T_I(y_1^{\#}) \leq x\}$ is a normal probability when $x > 0$, with mean zero and variance $(g^{\partial}(y_1^{\#})^+)^2 y_1^{\#}(1-y_1^{\#})\phi(y_1^{\#})^2$ for this normal distribution. For Case II ($L = L^*$ finite) and $0 < y \leq 1$ let

$$(4.15) \qquad T_{II}(y) = \begin{cases} 0 \text{ for } 0 < y < y_2^{\#} \\ \phi(y_2^{\#})g^{\partial}(y_2^{\#})^+\max(V(y_2^{\#}),0) \text{ for } y_2^{\#} \leq y \leq 1 \end{cases}$$

where $0 < y_2^{\#} < 1$. Thus, for y in $(y_2^{\#},1]$ we see that $T_{II}(y)$ is a normal random variable (having mean zero and variance $(g^{\partial}(y_2^{\#})^+)^2 y_2^{\#}(1-y_2^{\#})\phi(y_2^{\#})^2$ <u>but with the probability on the negative axis moved to the origin (zero)</u>. For Case III ($L > L^*$) we let

$$(4.16) \qquad T_{III}(y) = 0 \text{ for } 0 < y \leq 1.$$

Finally let

$$(4.17) \qquad t_\infty(y) = -\int_0^1 g^{\#}(s\wedge y)\phi'(s)ds, \text{ for } 0 < y \leq 1$$

which will also be finite by our assumptions.

One main goal is to show that as $n \to \infty$, the normalized bundle failure time $\sqrt{n}\,\{T_n - t_\infty(1)\}$ approaches $T_I(1)$ in distribution when $0 \leq L < L^*$, and $T_{II}(1)$ when $L = L^*$ (finite), and $T_{III}(1)$ when $L > L^*$. Recall that L^* is the initial collapse load defined in connection with Case II. Note that $t_\infty(1)$ is positive when $L < L^*$ and is zero when $L \geq L^*$. We are also interested in the time to failure of a given fraction y of fibers in the bundle. Letting [ny+1] be the integer part of ny+1, we will show that as $n \to \infty$ the normalized times $\sqrt{n}\,\{T_{[ny+1]} - t_\infty(y)\}$ approach $\mathcal{T}_{I}(y), \mathcal{T}_{II}(y)$, and $\mathcal{T}_{III}(y)$ in distribution for Cases I, II and III respectively. Note that $t_\infty(y) = 0$ for $0 < y \leq y_1^{\#}$ and $t_\infty(y) = t_\infty(1)$ for $y_2^{\#} \leq y \leq 1$. For the vast majority of cases these times to failure will be asymptotically normally distributed with mean $t_\infty(y)$ and variance $\sigma^2(y)$.

Our approach is as follows: First we demonstrate that the fiber failure times $T_1' \leq \ldots \leq T_n'$ for the quantile model of the previous section are equivalent in distribution to $T_1 \leq \ldots \leq T_n$ the fiber failure times for the bundle as originally described. This is accomplished in the following section. Hence, we need only show that as $n \to \infty$ the normalized bundle failure time $\mathcal{T}_n'(1)$ of the quantile model approaches $\mathcal{T}_I(1), \mathcal{T}_{II}(1)$ and $\mathcal{T}_{III}(1)$ in distribution for Cases I, II and III respectively. We also show the parallel results for $\mathcal{T}_n'(y)$ for fixed y in (0,1). Actually, we will prove the stronger convergence in probability when the uniform quantile processes V_n and the Brownian bridge V are constructed as in Pyke and Shorack (1968) and the appendix of Shorack (1972a). Our approach will parallel in some ways the approach used in the proofs of Shorack (1972a,1972b), and we will draw frequently on results given in the appendix of Shorack (1972a).

5. EQUIVALENCE IN DISTRIBUTION OF FAILURE TIMES FOR BOTH MODELS

The idea of using uniform random variables as building blocks for increasing sequences such as $\{T_i\}$ is well known. Hence, the equivalence in distribution of the random vectors $\{T_1,\ldots,T_n\}$ and $\{T_1',\ldots,T_n'\}$ is not surprising. We will only sketch the proof of equivalence, leaving the details to the reader.

Let $\underline{N}_0$ be the number of fibers which fail at time zero (as a result of the application of the bundle load program $\ell(t)$, $t \geq 0$). If all fibers do not fail immediately, that is, if $\underline{N}_0 < n$, let $\bar{T}_1 < \ldots < \bar{T}_M$ be the distinct time points at which the remaining $n-\underline{N}_0$ fail and let $\underline{N}_i \geq 1$ be the number of fibers which fail at time $\bar{T}_i$ for $i = 1, \ldots, M$. Either $\underline{N}_0 = n$ and $T_n = 0$ or $\sum_{i=0}^{M} \underline{N}_i = n$ and $\bar{T}_M = T_n > 0$.

Fix the time points $0 < t_1 < \ldots < t_m$ and the integers $\underline{n}_0, \underline{n}_1, \ldots, \underline{n}_m$ such that $\underline{n}_0 \geq 0$ and $\underline{n}_i \geq 1$ for $i = 1, \ldots, m$ and $\sum_{i=0}^{m} \underline{n}_i = n$. Let $\underline{dt}_i$ be the infinitesimal time interval $[t_i, t_i + dt_i)$ and let $\underline{dt}_i \otimes \underline{n}_i$ be the associated set in two-dimensional space of time and the nonnegative integers. Let $\underset{\sim}{t} = (t_1,\ldots,t_m)$ and $\underset{\sim}{n} = (\underline{n}_0,\ldots,\underline{n}_m)$ and let $A(\underset{\sim}{t},\underset{\sim}{n})$ be the event

$$(5.1) \qquad A(\underset{\sim}{t},\underset{\sim}{n}) = \{\underline{N}_0 = \underline{n}_0,\ (\bar{T}_1,\underline{N}_1) \in \underline{dt}_1 \otimes \underline{n}_1, \ldots, (\bar{T}_M,\underline{N}_M) \in \underline{dt}_m \otimes \underline{n}_m\} .$$

We wish to evaluate both $P\{A(\underset{\sim}{t},\underset{\sim}{n})\}$ and $P\{\underline{N}_0 = n\}$ for both the actual model and the quantile model where <u>primes on the random variables will be understood for the quantile model</u>. From the resulting expressions we will conclude that the quantile model is probabilistically equivalent to the real model.

First we define several quantities which will be useful in the proofs. Let $\bar{n}_j = \underline{n}_0 + \ldots + \underline{n}_j$ for $j = 0,\ldots,m$ so that $\bar{n}_j$ is the <u>number of failures up to and including time</u> t_j. Second let $n_i = \underline{n}_i - 1$ for $i = 1,\ldots,m$ and put $n_0 = \underline{n}_0$ so that n_i will be the additional fiber failures (Daniels feature) triggered at time t_i by the first fiber failure (Coleman feature) at time t_i. Note that

$\bar{n}_0 = n_0 = \underset{\sim}{n}_0$. We make use of the load histories $\ell_{n,0,0} = 0$,

(5.2) $$\ell_{n,0,j}(s) = n\ell(s)/(n-j+1) \text{ for } t \geq 0 \text{ and } j = 1,\dots,n_0 + 1$$

and

(5.3) $$\ell_{n,i,j}(s) = \begin{cases} n\ell(s)/(n-\bar{n}_{k-1}) \text{ for } t_{k-1} \leq t < t_k \text{ and } k = 1,\dots,i, \\ n\ell(s)/(n-\bar{n}_{i-1}-j) \text{ for } t_i \leq t \text{ and } j = 0,\dots,n_i + 1, \end{cases}$$

for $i = 1,\dots,m$ where $t_0 \equiv 0$. These are used in the generation of

(5.4) $$b_i(j) = F(t_i|\ell_{n,i,j})$$

for $i = 0,\dots,m$ and $j = 0,\dots,n_i + 1$. Finally, for $i = 0,\dots,m$ we let $H_i(s,0) = 1$ for $s \geq 0$ and

(5.5) $$H_i(s,r)$$

$$= \sum_{k=1}^{r} \sum_{1 \leq \bar{j}_1 < \dots < \bar{j}_k = r} \frac{s![b_i(1)-b_i(0)]^{j_1}[b_i(\bar{j}_1+1)-b_i(1)]^{j_2}\dots[b_i(\bar{j}_{k-1}+1)-b_i(\bar{j}_{k-2}+1)]^{j_k}}{j_1!\dots j_k!(s-r)!}$$

for $1 \leq r \leq s$, where $\bar{j}_v = j_1 + \dots + j_v$ and the sum is over $k = 1, \dots, r$ and all integers $1 \leq \bar{j}_1 < \dots < \bar{j}_k = r$.

<u>Lemma 1</u>: For the bundle as originally described and for the quantile model

(5.6) $$P\{A(\underset{\sim}{t},\underset{\sim}{n})\} = \prod_{j=1}^{m} (n-\bar{n}_{j-1})dF(t_j|\ell_{n,j,0}) \prod_{i=0}^{m} H_i(n-\bar{n}_{i-1}-1,n_i)$$

for $n_0 < n$ and $m \geq 1$. Also

(5.7) $$P\{\underset{\sim}{N}_0=n\} = P\{T_n=0\} = H_0(n,n).$$

Proof. For the bundle as originally described first examine the failure activity at time zero and compute $P\{\underset{\sim}{N}_0 = n_0\}$ for $0 \leq n_0 \leq n$. This is essentially as described by Daniels (1945). Next compute $P\{(\bar{T}_1,\underset{\sim}{N}_1) \in \underline{dt}_1 \otimes \underline{n}_1 | \underset{\sim}{N}_0 = \underline{n}_0\}$ when $1 \leq \underline{n}_1 \leq n-\bar{n}_0$. See Phoenix (1978) for part of this step and mimic the procedure at time zero for the remaining part. Similarly compute

$P\{(\bar{T}_{i+1},\underset{\sim}{N}_{i+1}) \in \underline{dt}_{i+1} \otimes \underline{n}_{i+1} | \underset{\sim}{N}_0=n_0;\ (\bar{T}_1,\underset{\sim}{N}_1) = (t_1,\underline{n}_1),\ldots,(\bar{T}_i,\underset{\sim}{N}_i) = (t_i,\underline{n}_i)\}$

for $\bar{n}_i < n$ and $1 \leq \underline{n}_{i+1} \leq n-\bar{n}_i$. Thus, to arrive at (5.6) we combine these expressions upon noting "given $(\bar{T}_i,\underset{\sim}{N}_i) \in \underline{dt}_i \otimes \underline{n}_i$" is the same as "given $(\bar{T}_i,\underset{\sim}{N}_i) = (t_i,\underline{n}_i)$". For the quantile model, the event $A(\underset{\sim}{t},\underset{\sim}{n})$ entails $T_1' = \ldots = T_{n_0}' = 0$, $T_{n_0+1}' = \ldots = T_{\bar{n}_1}' \in \underline{dt}_1,\ldots,T_{\bar{n}_{m-1}+1}' = \ldots = T_n' \in \underline{dt}_m$ when $1 \leq n_0 < n$. By (3.4) the V_i must satisfy

(5.8) $$V_j \leq F(0|\ell_{n,0,j}) = b_0(j) \text{ for } j = 1,\ldots,n_0$$

and for $i = 1,\ldots,m$

(5.9) $$F(t_i|\ell_{n,i,0}) \leq V_{\bar{n}_{i-1}+1} < F(t_i+dt_i|\ell_{n,i,0})$$

and

(5.10) $$V_{\bar{n}_{i-1}+j} \leq F(t_i|\ell_{n,i,j-1}) = b_i(j-1) \text{ for } j = 2,\ldots,\underline{n}_i \ .$$

Similar statements apply for $n_0 = 0$ and $n_0 = n$. Employing the well known joint distribution for $V_1 \leq \ldots \leq V_n$, which are an ordered sample from the uniform distribution, we again may construct (5.6) and (5.7).

<u>Theorem 1</u>:

The random vector $(T_1,\ldots,T_n)$ for the original model has the same distribution as the random vector $(T_1',\ldots,T_n')$ for the quantile model.

<u>Proof</u>. By our assumptions on m, and $t_1,\ldots,t_m$ and $\underline{n}_0,\ldots,\underline{n}_m$, all possible outcomes are represented by the events $A(\underset{\sim}{t},\underset{\sim}{n})$ and $\underline{N}_0 = n$. The theorem is immediate from Lemma 1.

We remark that results generated by Daniels (1945) for the static fiber bundle model follow from the results given here. In Daniels' problem, the bundle fails immediately under the constant load program $\ell_1(t) = L$, $t \geq 0$ or does not fail at all. Here, the event associated with immediate failure is the event $\{\underline{N}_0=n\}$ and its probability is $H_0(n,n)$. Under (1.2) we may view $\bar{F}(x) = \Psi(x,0)$ as the c.d.f. for the initial strength of the fibers and recall that $b_0(j) = \Psi(nL/(n-j+1))$ which is the probability that a given fiber will fail under the load $nL/(n-j+1)$ for $j = 1,\ldots,n$. Thus, $H_0(n,n)$ may be viewed as the c.d.f. for the initial bundle strength as a function of $L = L_s/n$ and is equivalent to $B_n(nL)$ of Daniels (1945), section 10, page 413.

We remark also that results generated by Phoenix (1978) for an earlier time dependent model follow from the results given here. In the earlier model $\Psi(x,z) = 1-\exp\{-\underset{\sim}{\Psi}(z)\}$, which is a function of z only so that $b_i(0) = \ldots = b_i(n_i+1)$ for $i = 1,\ldots,m$ and $b_0(0) = \ldots = b_0(n_0+1) = 0$. Hence, $H_i(s,r) = 0$ for $1 \leq r \leq s$ but $H_i(s,0) = 1$ for $i = 0,\ldots,m$. Consequently $P\{\underline{N}_0=n\} = 0$ and $P\{A(\underset{\sim}{t},\underset{\sim}{n})\}$ is nonzero only when $n_0 = n_1 = \ldots = n_m = 0$ which forces $m = n$ and $\bar{n}_1 = \ldots = \bar{n}_n = 1$. Thus, $A(\underset{\sim}{t},\underset{\sim}{n})$ is, in this case, the event $\{T_1 \in \underline{dt}_1, \ldots, T_n \in \underline{dt}_n\}$ and (5.6) becomes $P\{T_1 \in \underline{dt}_1,\ldots,T_n \in \underline{dt}_n\} = n! \prod_{i=1}^{n} dF(t_i|\ell_{n,i,0})$ which is equivalent to equation (2.14) of Phoenix (1978). Hence, fibers fail one-at-a-time in the earlier model.

Henceforth we drop the primes on the T_i's of the quantile model.

6. SUMMARY OF TECHNICAL ASSUMPTIONS

Earlier we introduced the functions Ψ^{-1}, g, g^{∂} and ϕ. Here we summarize earlier assumptions and properties and introduce further technical assumptions that will be required in the asymptotic analysis. Any new properties among those that follow are easily verified.

A1. On S the function $\Psi^{-1}(x,y)$ is nonnegative, is increasing in y, is decreasing in x, is jointly continuous in both variables and satisfies $\Psi^{-1}(x,0) = 0$.

A2. $\Psi^{-1}(x,y)$ is zero on S^0 and is positive and strictly increasing in y on S^+.

The partial derivatives of Ψ^{-1} are required to have special properties on $\bar{S}$.

A3. $\partial\Psi^{-1}(x,y)/\partial y$ and $\partial\Psi^{-1}(x,y)/\partial x$ are assumed to be jointly continuous in y and x on S^+.

A4. $\partial\Psi^{-1}(x,y)/\partial y$ and $\partial\Psi^{-1}(x,y)/\partial x$ are zero on S^0 by A2.

Let the bounding functions B and D be defined by

(6.1) $$B(y) = My^{-b_1}(1-y)^{-b_2} \text{ for } 0 < y < 1$$

and

(6.2) $$D(y) = My^{-3/2+b_1+\delta}(1-y)^{-3/2+b_2+\delta} \text{ for } 0 < y < 1,$$

where $M > 0$, $1/2 \leq b_1 < 1$, $b_2 \geq 1$ and $\delta > 0$ are constants. Also, L is a positive constant and we understand that M may depend on L.

A5. (Boundedness). We assume that $\partial\Psi^{-1}(x,y)/\partial y \leq B(y)$ for (x,y) in $\bar{S}$ and $x \geq L$.

A6. (Boundedness). Given $0 < b_3 < 1$ we assume $|\partial\Psi^{-1}(x,y)/\partial x| \leq M_{b_3}$ for (x,y) in $\bar{S}$ satisfying $0 < y \leq b_3$ and $L \leq x \leq L/(1-b_3)$ where M_{b_3} is a positive constant.

A7. (Case I). For points $0 \leq y_1^{\#} < y_2^{\#} \leq y_3^{\#} \leq 1$ we have that $g(y)$ is nonnegative and continuous on $[0,1)$, is zero on $[0,y_1^{\#})$ and on $[y_3^{\#},1)$, is strictly increasing on $[y_1^{\#},y_2^{\#})$ and is strictly decreasing on $[y_2^{\#},y_3^{\#})$.

A8. (Boundedness). We assume that $g'(y)$ exists and is continuous on $R = (0,y_1^{\#}) \cup (y_1^{\#}, y_3^{\#}) \cup (y_3^{\#}, 1)$, and that $g'(y) \leq B(y)$ on R.

A9. $g^{\partial}(y)$ is zero on $(0,y_1^{\#})$ and $(y_3^{\#},1)$, and is positive and continuous elsewhere on R.

B1. The function ϕ is bounded, continuous, decreasing and positive on $[0,1)$ and satisfies $\phi(y) \to 0$ as $y \to 1$.

B2. (Boundedness). We assume that the derivative ϕ' exists on $(0,1)$ and that $|\phi'(y)| \leq D(y)$.

The bounding of ϕ by D, and of g and $\partial\Psi^{-1}(x,y)/\partial y$ by B will permit us to derive important asymptotic results. These bounds are unrestrictive in most applications.

In the proofs that follow we use right continuous versions of some results in the Appendix of Shorack (1972a). These are as follows:

R1. (Convergence of V_n to V).

$$\sup_{0<y<1} |V_n(y)-V(y)| \to_e 0 , \tag{6.3}$$

that is, every sample path of the V_n process converges uniformly as $n\to\infty$ to the corresponding sample path of the V process.

R2. (Glivenko-Cantelli Lemma).

(6.4) $$\sup_{0\leq y\leq 1} |\Gamma_n^{-1}(y)-y| \rightarrow_e 0 \text{ as } n \rightarrow \infty .$$

R3. For $q(y) = [y(1-y)]^{1/2-\delta/2}$ we have

(6.5) $$\sup_{0<y<1} |V(y)|/q(y) = 0_p(1), \qquad \sup_{1/n\leq y\leq 1-1/n} |V_n(y)|/q(y) = 0_p(1)$$

for $n \geq 1$ and

(6.6) $$(n^{1/2}q(1/n))^{-1} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

R4. (Lemma A3 of Shorack (1972a)). Given ε there exists $0 < \beta = \beta_\varepsilon < 1$ and a subset $S_{n,\varepsilon}$ of Ω having $P(S_{n,\varepsilon}) > 1-\varepsilon$ on which

(6.7) $$\beta t \leq \Gamma_n^{-1}(t) \leq 1-\beta(1-t) \text{ for } 0 \leq t \leq 1.$$

Thus when B is our bounding function (6.1) we have

(6.8) $$B(\Gamma_n^{-1}(y)) \leq M_\beta B(y) \text{ for } 0 < y < 1$$

for some constant M_β.

We now proceed with the major theorems.

7. ASYMPTOTIC ANALYSIS OF THE QUANTILE MODEL

The main result of this section will be that given $0 < y \leq 1$ the sequence $\{T_n(y)\}_{n=1}^{\infty}$ converges in probability to a certain random variable when V_n and V are constructed as in Pyke and Shorack (1968). We begin by proving several lemmas.

Lemma 2: Given $0 < \varepsilon < 1$

(7.1)
$$\sup_{0 \leq y \leq 1-\varepsilon} |W_n(y) - g(y)| \to_e 0 \,,$$

(7.2)
$$\sup_{0 \leq y \leq 1-\varepsilon} |W_n^{\#}(y) - g^{\#}(y)| \to_e 0,$$

(7.3)
$$\sup_{0 \leq y \leq 1-\varepsilon} |g_n(y) - g(y)| \to 0 \,,$$

and

(7.4)
$$\sup_{0 \leq y \leq 1-\varepsilon} |g_n^{\#}(y) - g^{\#}(y)| \to 0 \text{ as } n \to \infty \,.$$

Proof: By assumption A1 we have uniform continuity of Ψ^{-1} on the compact set $\{(x,y);\ L \leq x \leq L/(1-\varepsilon/2) \text{ and } 0 \leq y \leq 1-\varepsilon/2\}$. Now $\sup\{|L/(1-y) - L/(1-\gamma_n(y))|;\ 0 \leq y \leq 1-\varepsilon\} \to 0$ as $n \to \infty$ since $0 \leq y - \gamma_n(y) \leq 1/n$ for y in $[0,1]$. Also $\sup\{|\Gamma_n^{-1}(y)-y|;\ \ 0 \leq y \leq 1\} \to_e 0$ as $n \to \infty$ by (6.4). Thus,

$$\sup_{0 \leq y \leq 1-\varepsilon} |\Psi^{-1}(L/(1-\gamma_n(y)),\ \Gamma_n^{-1}(y)) - \Psi^{-1}(L/(1-y),y)| \to_e 0$$

and (7.1) is verified. Similarly, (7.3) is verified. Now for $0 \leq y \leq 1-\varepsilon$ we have

$$\left| \sup_{0\leq\tau\leq y} W_n(\tau) - \sup_{0\leq\tau\leq y} g(\tau) \right| \leq \sup_{0\leq\tau\leq y} |W_n(\tau) - g(\tau)|$$

and (7.2) is verified. Similarly, (7.4) is verified.

Lemma 3: Let y be in R and let

$$R_\delta(y) = \{x;\ |x-y| \leq \delta \text{ and } 0 \leq x \leq 1\}. \tag{7.5}$$

If $\delta > 0$ is chosen so that $R_\delta(y) \subset R$ then as $n \to \infty$

$$\sup\{|Z_n(x)-Z(x)|;\ x \in R_\delta(y)\} \to_e 0. \tag{7.6}$$

Proof: Let

$$A_n(x) = \{\Psi^{-1}(L/(1-\gamma_n(x)),\ \Gamma_n^{-1}(x)) - g_n(x)\}/(\Gamma_n^{-1}(x) - x), \tag{7.7}$$

(taking the right limit at points x where $\Gamma_n^{-1}(x) = x$). We have $Z_n(x) = A_n(x)V_n(x)$ and $Z(x) = g^\partial(x)V(x)$ on R and

$$\begin{aligned} |Z_n-Z| &\leq |A_nV_n-A_nV| + |A_nV-g^\partial V| \\ &\leq |A_n||V_n-V| + |V||A_n-g^\partial|. \end{aligned} \tag{7.8}$$

Now $\sup\{|V_n(x)-V(x)|;\ 0 < x < 1\} \to_e 0$ and $V(x)$ is bounded on $(0,1)$ for fixed ω in Ω. The lemma will follow upon verifying that as $n \to \infty$

$$(7.9) \qquad \sup\{|A_n(x)-g^\partial(x)|;\ x \in R_\delta(y)\} \to_e 0$$

since $g^\partial(x)$ is bounded on $R_\delta(y)$. By assumptions A3 and A4 we have continuity of $\partial\Psi^{-1}(\eta,\xi)/\partial\xi$ on the set $\{(\eta,\xi);\ x \in R_\delta(y)$ and $|\xi-x| \leq \delta'$ and $|L/(1-x)-\eta| \leq \delta''\}$ where $\delta' > 0$ and $\delta'' >$ are chosen so that this set is contained in $\bar{S}$ (as can be done by the line touching assumptions) and where $R_{\delta+\delta'}(y) \subset R$ since R is open. Moreover, the continuity is uniform since the set is compact. Now $\sup\{|L/(1-x) - L/(1-\gamma_n(x))|;\ x \in R_\delta(y)\} \to 0$ as $n \to \infty$ and we have (6.4) also. Thus, for n exceeding some $n_{y,\delta,\delta',\delta'',\omega}$ we have for x in $R_\delta(y)$

$$(7.10) \quad |A_n(x)-g^\partial(x)| \leq \left|\frac{\partial\Psi^{-1}}{\partial\xi}(L/(1-\gamma_n(x)),\ \xi) - \frac{\partial\Psi^{-1}}{\partial z}(L/(1-x),\ z)\Big|_{z=x}\right|$$

for some ξ between $\Gamma_n^{-1}(x)$ and x, and (7.9) follows from the uniform continuity.

Lemma 4: Let

$$(7.11) \qquad R_\delta^\#(y) = \{x;\ y^\#(y) - \delta \leq x \leq y^\#(y-\delta) + \delta \ \text{ and } \ 0 \leq x \leq 1\}.$$

Fix ω in Ω, y in $(0,1)$ and $\delta > 0$ such that $R_\delta^\#(y) \subset (0,1)$. For n exceeding some $n_{\delta,\omega,y}$ we have

$$(7.12) \qquad W_n^\#(y) = \sup\{W_n(x);\ x \in R_\delta^\#(y)\},$$

and for n exceeding some $n_{\delta,y}$

$$(7.13) \qquad g_n^{\#}(y) = \sup\{g_n(x);\ x \in R_\delta^{\#}(y)\}.$$

<u>Proof</u>: Suppose $0 < y \leq y_1^{\#}$. Then by assumptions A1 and A2 we have $\Psi^{-1}(\eta,\xi) = 0$ on the set $\{(\eta,\xi);\ 0 \leq x \leq y-\delta$ and $|L/(1-x)-\eta| \leq \delta'$ and $0 \leq \xi \leq y - \delta/2\}$ for some $\delta' > 0$ depending on δ. Now for $0 < y \leq y_1^{\#}$ we have

$$(7.14) \qquad \sup\{W_n(x);\ 0 \leq x \leq y\}$$

$$= \max[\sup\{W_n(x);\ 0 \leq x \leq y-\delta\},\ \sup\{W_n(x);\ x \in R_\delta^{\#}(y)\}]$$

and by (6.4) we have $\sup\{|\Gamma_n^{-1}(x)-x|;\ 0 \leq x \leq y-\delta\} < \delta/2$ for n exceeding some $n_{\delta,\omega}$. Also, $\sup\{|L/(1-x) - L/(1-\gamma_n(x))|;\ 0 \leq x \leq y-\delta\} < \delta'$ for n exceeding some n_δ. Recalling (3.9) we have for n exceeding $n'_{\delta,\omega} \equiv \max\{n_{\delta,\omega}, n_\delta\}$ that

$$(7.15) \qquad \sup\{W_n(x);\ 0 \leq x \leq y-\delta\} = 0.$$

But $W_n(y) \geq 0$ for $0 \leq y \leq 1$ and (7.12) follows from (7.14). Next suppose $y_1^{\#} < y \leq y_2^{\#}$. Then by assumption A7 we have $g(y) > 0$ and $g(y)$ strictly increasing on $[y_1^{\#}, y_2^{\#})$. Thus $g(y) - g(y-\delta) \geq \delta''$ for some $\delta'' > 0$ depending on y. By Lemma 2, for n exceeding some $n_{\delta,\omega,y}$ we have $\sup\{W_n(x);\ 0 \leq x \leq y-\delta\} < g(y-\delta) + \delta''/2 \leq g(y) - \delta''/2$ while $W_n(y) > g(y) - \delta''/2$. Now $y^{\#}(y) = y$ so that $R_\delta^{\#}(y) = \{x;\ y-\delta \leq x \leq y\}$. Hence, (7.12) easily follows. Lastly suppose that $y_2^{\#} < y < 1$. Then we already have

$$(7.16) \qquad \sup\{W_n(x);\ 0 \le x \le y\} = \sup\{W_n(x);\ y_2^{\#} - \delta \le x \le y\}$$

for n exceeding some $n_{\delta,\omega}$ and (7.12) is verified for $y_2^{\#} < y \le y_2^{\#} + \delta$. If $y_2^{\#} + \delta < y < 1$ we have by assumption A7 that $g(y_2^{\#}) - g(x) > \delta'''$ for $y_2^{\#} + \delta \le x \le y$ and some $\delta''' > 0$. By Lemma 2 we have $\sup\{W_n(x);\ y_2^{\#} + \delta \le x \le y\} < (g(y_2^{\#}) + g(y_2^{\#} + \delta))/2$ while $W_n(y_2^{\#}) > (g(y_2^{\#}) + g(y_2^{\#} + \delta))/2$ for n exceeding some $n_{\delta,\omega,y}$ so that (7.12) holds again. The proof of (7.13) is analagous.

Lemma 5: (Case I) Suppose y is in $(0,y_1^{\#}) \cup (y_1^{\#},1)$. Then as $n \to \infty$,

$$(7.17) \qquad |Z_n^{\#}(y) - Z(y^{\#}(y))| \to_e 0.$$

Proof: Fix $\varepsilon > 0$ and ω in Ω and y in $(0,y_1^{\#}) \cup (y_1^{\#},1)$. Then there exists $\delta > 0$ such that $R_\delta(y^{\#}(y)) \subset R$ and

$$(7.18) \qquad \sup\{|Z(x) - Z(y^{\#}(y))|;\ x \in R_\delta(y^{\#}(y))\} \le \varepsilon/6.$$

This follows since $Z(x) = g^{\partial}(x)V(x)$ is continuous in R given ω in Ω. By Lemma 3 we also have for n exceeding some $n_{\varepsilon,\delta,\omega,y}$ that

$$(7.19) \qquad \sup\{|Z_n(x) - Z(x)|;\ x \in R_\delta(y^{\#}(y))\} \le \varepsilon/6.$$

Now $|Z_n(x) - Z_n(y)| \le |Z_n(x) - Z(x)| + |Z_n(y) - Z(y)| + |Z(x) - Z(y)|$. Hence, by (7.18) and (7.19) we have for $n > n_{\varepsilon,\delta,\omega,y}$ that

$$(7.20) \qquad \sup\{|Z_n(x) - Z_n(y^{\#}(y))|;\ x \in R_\delta(y^{\#}(y))\} \le \varepsilon/2.$$

We recall from Lemma 4 that for n exceeding some $n'_{\delta,\omega,y}$ we have $\sup\{W_n(x);\ 0 \le x \le y\} = \sup\{W_n(x);\ x \in R^{\#}_{\delta}(y)\}$ and for n exceeding some $n'_{\delta,y}$ we have $\sup\{g_n(x);\ 0 \le x \le y\} = \sup\{g_n(x);\ x \in R^{\#}_{\delta}(y)\}$. Hence, for $n > n_{\delta,\omega,y} \equiv \max\{n'_{\delta,y},\ n'_{\delta,\omega,y}\}$ we have

$$(7.21)\qquad Z^{\#}_n(y) = \sqrt{n}\,[\sup\{W_n(x);\ x \in R^{\#}_{\delta}(y)\} - \sup\{g_n(x);\ x \in R^{\#}_{\delta}(y)\}].$$

Also, $$R^{\#}_{\delta}(y) \subset R_{\delta}(y^{\#}(y)) \subset R \text{ and}$$

$$\inf\{W_n(x)-g_n(x);\ x \in R^{\#}_{\delta}(y)\}$$

$$(7.22)\qquad \le \sup\{W_n(x);\ x \in R^{\#}_{\delta}(y)\} - \sup\{g_n(x);\ x \in R^{\#}_{\delta}(y)\}$$

$$\le \sup\{W_n(x) - g_n(x);\ x \in R^{\#}_{\delta}(y)\}.$$

Thus, for $n > n_{\delta,\omega,y}$ we have

$$(7.23)\qquad \inf\{Z_n(x);\ x \in R_{\delta}(y^{\#}(y))\} \le Z^{\#}_n(y) \le \sup\{Z_n(x);\ x \in R_{\delta}(y^{\#}(y))\}.$$

Now by (7.20) we have both

$$|\sup\{Z_n(x);\ x \in R_{\delta}(y^{\#}(y))\} - Z_n(y^{\#}(y))| \le \varepsilon/2$$

and

$$|\inf\{Z_n(x);\ x \in R_{\delta}(y^{\#}(y))\} - Z_n(y^{\#}(y))| \le \varepsilon/2$$

for n exceeding some $n_{\varepsilon,\delta,\omega,y}$. Taking $n'_{\varepsilon,\delta,\omega,y}$ $\max\{n_{\delta,\omega,y}, n_{\varepsilon,\delta,\omega,y}\}$ we have for $n > n'_{\varepsilon,\delta,\omega,y}$ that

$$|Z^{\#}_n(y) - Z_n(y^{\#}(y))| \leq \varepsilon/2 \tag{7.24}$$

in view of (7.23) Observe that

$$|Z^{\#}_n(y) - Z(y^{\#}(y))| \leq |Z^{\#}_n(y) - Z_n(y^{\#}(y))| + |Z_n(y^{\#}(y)) - Z(y^{\#}(y))|.$$

Hence, for $n > n'_{\varepsilon,\delta,\omega,y}$ we have from (7.24) and (7.19) that

$$|Z^{\#}_n(y) - Z(y^{\#}(y))| \leq \varepsilon/2 + \varepsilon/6 \leq \varepsilon \tag{7.25}$$

which gives us the lemma.

<u>Lemma 6</u>. (Case I) Suppose $0 < y^{\#}_1 < 1$. Then as $n \to \infty$,

$$|Z^{\#}_n(y^{\#}_1) - g^{\partial}(y^{\#}_1)^{+}\max(V(y^{\#}_1), 0)| \underset{\text{a.e.}}{\to} 0. \tag{7.26}$$

<u>Proof:</u> Take ω in Ω so that $V(y^{\#}_1) < 0$. Then since $V(y)$ is continuous on $[0,1]$ there exists δ satisfying $0 < \delta < y^{\#}_1$ such that $V(y) < 0$ for $y^{\#}_1 - \delta \leq y \leq y^{\#}_1$. By (6.3) we have for n exceeding some $n_{\delta,\omega}$ that $V_n(y) < 0$ and thus $\Gamma^{-1}_n(y) < y$ for $y^{\#}_1 - \delta \leq y \leq y^{\#}_1$. Now $\Psi^{-1}(x,y)$ is increasing in y on S by A1. Hence, by (3.9) and (3.16) we have

$$0 \leq W_n(y) \leq g_n(y) \leq \Psi^{-1}\left(\frac{L}{1-\gamma_n(y)}, y\right) - \Psi^{-1}\left(\frac{L}{1-\gamma_n(y)}, \gamma_n(y)\right) \tag{7.27}$$

for y in $[y_1^\# - \delta, y_1^\#]$ where we have subtracted the quantity $g(\gamma_n(y))$ which is zero since $\gamma_n(y) \le y_1^\#$. By assumption A5, $\partial\Psi^{-1}(x,y)/\partial y \le B(y)$ for (x,y) in $\bar{S}$ and $x \ge L$. Thus, upon observing the properties of the bounding function $B(y)$ we obtain from (7.27)

$$0 \le W_n(y) \le g_n(y) \le (B(\gamma_n(y_1^\# - \delta)) + B(y_1^\#))(y - \gamma_n(y)), \tag{7.28}$$

for y in $[y_1^\# - \delta, y_1^\#]$. Now we have $0 \le y - \gamma_n(y) \le 1/n$ for y in $[0,1]$ and from the proof of Lemma 4 we know that $W_n^\#(y_1^\#-\delta) = 0$ and $g_n^\#(y_1^\#-\delta) = 0$ for n exceeding some $n'_{\delta,w}$. Also,

$$g_n^\#(y_1^\#) = \max[g_n^\#(y_1^\#-\delta), \sup_{0\le z\le\delta} g_n(y_1^\#-\delta+z)] ,$$

and thus it follows from (7.28) that $\sqrt{n}\, g_n^\#(y_1^\#) \to 0$ as $n \to \infty$. Similarly, $\sqrt{n}\, W_n^\#(y_1^\#) \to 0$ as $n \to \infty$. Thus $Z_n^\#(y_1^\#) \to 0$ as $n \to \infty$.

Next suppose $\varepsilon > 0$ is given and ω in Ω is given such that $V(y_1^\#) > 0$. Since $V(y)$ is continuous on $[0,1]$ there exists a δ depending on ε and satisfying $0 < \delta < y_1^\#$ such that $V(y) > 0$ for $y_1^\# - \delta \le y \le y_1^\#$ and

$$g^\partial(y_1^\#) + \sup\{|V(y) - V(y_1^\#)|;\ y_1^\# - \delta \le y \le y_1^\#\} \le \frac{\varepsilon}{2}. \tag{7.29}$$

Also, since $V(y_1^\#) > 0$ we have by (6.3) that $\Gamma_n^{-1}(y) > y$ for $y_1^\# - \delta \le y \le y_1^\#$ and n exceeding some $n_{\delta,\omega}$. Now for $n > n_{\delta,\omega}$

$$W_n(y_1^\#) = \Psi^{-1}\left(\frac{L}{1-\gamma_n(y_1^\#)}\,,\ \Gamma_n^{-1}(y_1^\#)\right) \tag{7.30}$$

$$= \frac{\partial\Psi^{-1}}{\partial\xi}\left(\frac{L}{1-\gamma_n(y_1^\#)}\,,\ \xi\right)(\Gamma_n^{-1}(y_1^\#) - y_1^\#) + g_n(y_1^\#)$$

for some $y_1^\# < \xi \leq \Gamma_n^{-1}(y_1^\#)$, and note that the point $(L/(1-\gamma_n(y_1^\#)), \xi)$ is in S^+. Hence, for $n > n_{\delta,\omega}$

(7.31) $$\sqrt{n}\, W_n(y_1^\#) = \frac{\partial\Psi^{-1}}{\partial\xi}\left(\frac{L}{1-\gamma_n(y_1^\#)}, \xi\right) V_n(y_1^\#) + \sqrt{n}\, g_n(y_1^\#).$$

On the other hand for y in $[y_1^\# - \delta, y_1^\#]$ we have

(7.32) $$\sqrt{n}\, W_n(y) \leq \frac{\partial\Psi^{-1}}{\partial\eta}(\zeta,\eta)V_n(y) + \sqrt{n}\, g_n(y)$$

for some point (ζ,η) in the set

$$S^+ \cap \{(x,z);\ y_1^\# - \delta \leq z \leq \Gamma_n^{-1}(y_1^\#),\ L/(1-\gamma_n(y_1^\#-\delta)) \leq x \leq L/(1-y_1^\#)\}$$

which is nonempty for all $n > n_{\delta,w}$. Now by (6.3) we have $\sup\{|V_n(y)-V(y)|;$ $y_1^\#-\delta \leq y \leq y_1^\#\} \to 0$ for fixed ω in Ω and note that $V(y)$ is bounded. Moreover, by the joint continuity (A3) and $B(y)$ bounding (A5) of $\partial\Psi^{-1}(x,y)/\partial y$ we may shrink δ and choose n sufficiently large so that both $\partial\Psi^{-1}/\partial\xi$ of (7.31) and $\partial\Psi^{-1}/\partial\eta$ of (7.32) are as close to $g^\partial(y_1^\#)^+$ as we wish. Thus, we may shrink δ and choose $n_{\omega,\varepsilon}$ sufficiently large so that

(7.33) $$|(\partial\Psi^{-1}/\partial\xi)V_n(y_1^\#)-g^\partial(y_1^\#)^+V(y_1^\#)| \leq \varepsilon/8$$

for (7.31), and

(7.34) $$|(\partial\Psi^{-1}/\partial\eta)V_n(y)-g^\partial(y_1^\#)^+V(y)| \leq \varepsilon/8$$

for (7.32) for all n exceeding $n_{\omega,\varepsilon}$. Recall also that $0 \leq \sqrt{n}\, g_n^\#(y_1^\#) = \sup\{\sqrt{n}\, g_n(y);$ $0 \leq y \leq y_1^\#\} \leq \varepsilon/8$ for all n exceeding some n_ε. Thus, from (7.31) we obtain

$$(7.35) \qquad |\sqrt{n}\ W_n(y_1^\#)-g^\partial(y_1^\#)^+V(y_1^\#)| \leq \varepsilon/4$$

for all n exceeding $n'_{\omega,\varepsilon} \equiv \max\ [n_\varepsilon, n_{\omega,\varepsilon}]$. Similarly, for all y in $[y_1^\#-\delta, y_1^\#]$ we find from (7.32)

$$(7.36) \qquad \sqrt{n}\ W_n(y) \leq g^\partial(y_1^\#)^+V(y) + \varepsilon/4$$

for n exceeding $n'_{\omega,\varepsilon}$. Now from before $\sqrt{n}\ W_n^\#(y_1^\#-\delta) = 0$ for $n > n'_{\delta,\omega}$ and we have

$$(7.37) \qquad W_n^\#(y_1^\#) = \max[W_n^\#(y_1^\#-\delta), \sup_{0\leq z\leq\delta} W_n(y_1^\#-\delta+z)]\ .$$

Thus, by (7.29) and (7.35) to (7.37) we arrive at the key relation

$$(7.38) \qquad \sqrt{n}\ W_n(y_1^\#) \leq \sqrt{n}\ W_n^\#(y_1^\#) \leq \sqrt{n}\ W_n(y_1^\#) + \varepsilon$$

for n exceeding some $n''_{\omega,\varepsilon} \equiv \max[n'_{\delta,\omega}, n'_{\omega,\varepsilon}]$. Since $Z_n^\#(y_1^\#) = \sqrt{n}\ \{W_n^\#(y_1^\#)-g_n^\#(y_1^\#)\}$ and $\sqrt{n}\ g_n^\#(y_1^\#) \to 0$ as $n \to \infty$ we obtain from (7.38) that $Z_n^\#(y_1^\#) \to Z_n(y_1^\#)$ and from (7.35) that $Z_n(y_1^\#) \to g^\partial(y_1^\#)^+V(y_1^\#)$ for this ω in Ω . The set ω for which $V(y_1^\#) = 0$ has measure zero and our result is a.e. and the proof is complete.

Lemma 7: (Case II). Under the assumptions of Case II,

(7.39) $$Z_n^{\#}(y) \to_e 0 \text{ for } y \text{ in } [0, y_2^{\#})$$

and

(7.40) $$|Z_n^{\#}(y) - g^{\partial}(y_2^{\#})^{+}\max(V(y_2^{\#}),0)| \underset{a.e.}{\to} 0 \quad \text{for } y \text{ in } [y_2^{\#}, 1).$$

Proof: The proof is similar to that of Lemma 6. To show (7.39) we recall that $y_1^{\#} = y_2^{\#}$ and from the proof of Lemma 4, that $W_n^{\#}(y_1^{\#}-\delta) = 0$ and $g_n^{\#}(y_1^{\#}-\delta) = 0$ for $\delta > 0$ and n exceeding some $n_{\delta,\omega}$. Hence choose $\delta > 0$ such that $y < y_1^{\#} - \delta$. To verify (7.40) we repeat the proof of Lemma 6 with the following modifications. Replace $y_1^{\#} - \delta \leq y \leq y_1^{\#}$ with $y_2^{\#} - \delta \leq y' \leq y_2^{\#} + \delta$ and make the other associated changes. Now for n exceeding some $n_{\delta,\omega,y}$ we have from Lemma 4 that $W_n^{\#}(y) = \sup\{W_n(x);\ x \in R_\delta^{\#}(y)\}$ and $g_n^{\#}(y) = \sup\{g_n(x);\ x \in R_\delta^{\#}(y)\}$. Hence when ω is in Ω such that $V(y_2^{\#}) < 0$, we obtain $Z_n^{\#}(x) \to 0$ as $n \to \infty$. For ω in Ω such that $V(y_2^{\#}) > 0$ we find $|\sqrt{n}\, g_n^{\#}(y_2^{\#} + \delta)| \to 0$ as $n \to \infty$ and arrive at the analog of (7.38) which is

(7.41) $$\sqrt{n}\, W_n(y_2^{\#}) \leq \sqrt{n}\, W_n^{\#}(y) \leq \sqrt{n}\, W_n(y_2^{\#}) + \varepsilon$$

whence $Z_n^{\#}(y) \to Z_n^{\#}(y_2^{\#}) \to g^{\partial}(y_2^{\#})^{+}V(y_2^{\#})$ as $n \to \infty$. Again the set of ω for which $V(y_2^{\#}) = 0$ has measure zero and the proof is complete.

Lemma 8: (Case III). Suppose y is in $(0,1)$. Then as $n \to \infty$

$$|Z_n^{\#}(y)| \underset{e}{\to} 0. \tag{7.42}$$

Proof: Choose $\varepsilon > 0$ such that $y < 1-\varepsilon$. Then by A1 and A2, $\Psi^{-1}(\eta,\xi) = 0$ on the set $\{(\eta\ \xi);\ 0 \le x \le 1-\varepsilon$ and $|L/(1-x)-\eta| \le \delta$ and $0 \le \xi \le 1-\varepsilon/2\}$ for some $\delta > 0$ depending on ε. By (6.4) and the fact that $\sup\{|L/(1-x)-L/(1-\gamma_n(x))|;\ 0 \le x \le 1-\varepsilon\} \to 0$ as $n \to \infty$ we have from (3.9) that $\sup\{W_n(x);\ 0 \le x \le 1-\varepsilon\} = 0$ for n exceeding some $n_{\varepsilon,\omega}$. Similarly, $\sup\{g_n(x);\ 0 \le x \le 1-\varepsilon\} = 0$ for n exceeding some n_ε. Hence, (7.42) is immediate.

Lemma 9: Let

$$R_{n1} = \sqrt{n}\,(\phi(1/n)-\phi(0))W_1\ , \tag{7.43}$$

$$R_{n2} = \sqrt{n}\ \phi((n-1)/n)\max\{W_i;\ i = 1,\ldots,n\}\ , \tag{7.44}$$

$$R_{n3} = \sqrt{n}\int_0^{1/n}\phi'(y)\Psi^{-1}(L,y)dy\ , \tag{7.45}$$

and

$$R_{n4} = \sqrt{n}\int_{(n-1)/n}^{1}\phi'(y)\Psi^{-1}(L,y)dy. \tag{7.46}$$

Then as $n \to \infty$

$$R_{n1} \underset{p}{\to} 0,\ R_{n2} \to_p 0,\ R_{n3} \to 0,\ R_{n4} \to 0. \tag{7.47}$$

<u>Proof</u>: For R_{n1} we have $W_1 = \Psi^{-1}(L,V_1)$ and by A5 and B2 we may write

$$(7.48) \qquad |R_{n1}| \le \sqrt{n}\,(\int_0^{V_1} B(y)dy)(\int_0^{1/n} D(y)dy).$$

On the event $V_1 < 1/2$ and for $n > 2$ we have

$$(7.49) \qquad |R_{n1}| \le \sqrt{n}\, M_1 V_1^{-b_1+1}(1/n)^{b_1+\delta-1/2} = M_1(nV_1)^{-b_1+1} n^{-\delta}.$$

Now $P(V_1 < 1/2) \to 1$ as $n \to \infty$ and $nV_1 = O_p(1)$. Hence, since $\delta > 0$ we obtain $|R_{n1}| \le M_1 n^{-\delta} O_p(1) \xrightarrow{p} 0$ as $n \to \infty$. Next, consider R_{n2} and observe by A1 that $0 \le \max\{W_i;\ i = 1,\dots,n\} \le \Psi^{-1}(L,V_n)$. Hence, on the event $V_n > 1/2$ and for $b_2 \ne 1$ we have

$$|R_{n2}| \le \sqrt{n}\,|\int_{1-1/n}^{1} D(y)dy|\,|\int_0^{V_n} B(y)dy|$$

$$(7.50) \qquad \le \sqrt{n}\, M_2\,(\tfrac{1}{n})^{-1/2+b_2+\delta}\,|M_2' + M_2''(1-V_n)^{-b_2+1}|$$

$$\le M_2 M_2' n^{-b_2-\delta+1} + M_2 M_2'' n^{-b_2-\delta+1}(1-V_n)^{-b_2+1}.$$

Recall that $(1-V_n)n = O_p(1)$ and in fact has a limiting nondegenerate continuous distribution as $n \to \infty$ and hence so does $((1-V_n)n)^{-b_2+1}$. Now $b_2 \ge 1$ and $P(V_n > 1/2) \to 1$ as $n \to \infty$. Thus $M_2 M_2' n^{-b_2-\delta+1} \to 0$ as $n \to \infty$ and $M_2 M_2'' n^{-\delta}((1-V_n)n)^{-b_2+1} = n^{-\delta} O_p(1) \to_p 0$ as $n \to \infty$. Thus $|R_{n2}| \xrightarrow{p} 0$ for $b_2 \ne 1$. If $b_2 = 1$, the last term in (7.50) is instead $- M_2 M_2''(1/n)^{\delta}\ (\log(n(1-V_n)) - \log n)$, and we arrive at the same conclusion.

For R_{n3} we have for $n > 1$

$$|R_{n3}| \le \sqrt{n}\ \left|\int_0^{1/n} \left|\int_0^y B(x)dx\right| D(y)dy\right|$$

(7.51) $$\le M_3 \sqrt{n} \int_0^{1/n} y^{-b_1+1} y^{-3/2+b_1+\delta} dy = M_3' n^{-\delta} \to 0$$

as $n \to \infty$. Finally, for R_{n4} we have for $n > 1$ and $b_2 > 1$,

$$|R_{n4}| \le \sqrt{n}\ M_4 \left|\int_{1-1/n}^1 (1-y)^{b_2-3/2+\delta} (M_4' + M_4'' \int_0^y (1-x)^{-b_2} dx)dy\right|$$

(7.52) $$\le \sqrt{n}\ M_4 \left|\int_{1-1/n}^1 (1-y)^{b_2-3/2+\delta} (M_4''' + M_4''''(1-y)^{-b_2+1})dy\right|$$

$$\le 2\sqrt{n}\ M_4 M_4'''(1/n)^{b_2+\delta-1/2} + 2\sqrt{n}\ M_4 M_4''''(1/n)^{\delta+1/2}$$

$$\to 0$$

as $n \to \infty$. If $b_2 = 1$, the last term in (7.52) becomes instead $\sqrt{n}\ M_4 M_4'' \int_0^{1/n} z^{-1/2+\delta} \log(1/z)dz$ and we arrive at the same conclusion.

Before proving the major theorems of this section, we prove one more lemma. In what follows the "*" restriction will mean $f^*(t) = f(t)$ for $1/n \leq t \leq 1-1/n$, and $f^*(t) = 0$ otherwise for f on (0,1) or [0,1].

<u>Lemma 10</u>: Given $\varepsilon > 0$ there exist subsets $\bar{S}_{n,\varepsilon}$, $n \geq 1$ satisfying $P(\bar{S}_{n,\varepsilon}) > 1-\varepsilon$ for which

$$(7.53) \qquad \chi(\bar{S}_{n,\varepsilon})|Z_n^{\#*}(s)| \leq \bar{M}_\varepsilon B(s)q(s) \text{ for s in } (0,1)$$

where $\chi(S)$ is the indicator function for the set S, $\bar{M}_\varepsilon$ is some constant, and q(s) is as defined in result R3.

<u>Proof</u>: Let b be a constant satisfying $0 < b < y_2^{\#}$ and suppose s is in (0,b]. From A6 (take $b_3 = (b+1)/2$) and (6.4) and the fact that $0 \leq y - \gamma_n(y) \leq 1/n$, we have

$$(7.54) \qquad |\partial\Theta(\eta,s)/\partial\eta| \leq M_b' \text{ where } \Theta(\eta,s) = \Psi^{-1}(L/(1-\eta), \ \Gamma_n^{-1}(s))$$

for some constant M_b' when n exceeds some $n_{b,\omega}$, when η lies between $\min(\Gamma_n^{-1}(s), \gamma_n(s))$ and $\max(\Gamma_n^{-1}(s), s)$ and when $(L/(1-\eta), \Gamma_n^{-1}(s))$ is in $\bar{S}$. Thus, by (4.5),(3.9) and (7.54) we obtain

$$(7.55) \quad g(\Gamma_n^{-1}(s)) - M_b'|\gamma_n(s)-\Gamma_n^{-1}(s)| \leq W_n(s) \leq g(\Gamma_n^{-1}(s)) + M_b'|\gamma_n(s)-\Gamma_n^{-1}(s)|$$

for $n > n_{b,\omega}$. From (7.55) we obtain

$$\begin{aligned} & -M_b' \sup\{|\Gamma_n^{-1}(\tau)-\gamma_n(\tau)|;\ 0 \leq \tau \leq s\} \\ (7.56) \qquad & \leq W_n^{\#}(s) - g^{\#}(\Gamma_n^{-1}(s)) \\ & \leq M_b' \sup\{|\Gamma_n^{-1}(\tau)-\gamma_n(\tau)|;\ 0 \leq \tau \leq s\} \end{aligned}$$

whence

$$|Z_n^{\#}(s)| \le \sqrt{n}\ |g^{\#}(\Gamma_n^{-1}(s))-g^{\#}(s)| + \sqrt{n}\ |g^{\#}(s)-g_n^{\#}(s)|$$

(7.57)

$$+ M_b' \sup\{|V_n(\tau)|;\ 0 \le \tau \le s\} + M_b' \sup\{|\sqrt{n}\ (\gamma_n(\tau)-\tau)|;\ 0 \le \tau \le s\}.$$

Using (7.54) again we obtain in a similar fashion

$$\sqrt{n}\ |g^{\#}(s)-g_n^{\#}(s)| \le M_b' \sup\{|\sqrt{n}\ (\gamma_n(\tau)-\tau)|;\ 0 \le \tau \le s\} \tag{7.58}$$

for $n > n_{b,\omega}$. Now, by A8 we have $g'(\eta) \le B(s) + B(\Gamma_n^{-1}(s))$ for η between s and $\Gamma_n^{-1}(s)$ and η in R since $b_1 > 0$ and $b_2 > 0$. Furthermore $g(s)$ is increasing on $[0, y_2^{\#})$. Thus, by (6.4) we obtain for n exceeding some $n_{b,\omega}'$ and for s in $(0,b]$

$$\sqrt{n}\ |g^{\#}(\Gamma_n^{-1}(s))-g^{\#}(s)| = \sqrt{n}\ |g(\Gamma_n^{-1}(s))-g(s)|$$

(7.59)

$$\le (B(s) + B(\Gamma_n^{-1}(s)))|V_n(s)|.$$

Consequently, for s in $(0,b]$ and $n > n_{b,\omega}'' = \max\{n_{b,\omega}, n_{b,\omega}'\}$ we have the first important bound

$$|Z_n^{\#}(s)| \le (B(s) + B(\Gamma_n^{-1}(s)))|V_n(s)|$$

$$+ 2M_b' \sup\{|\sqrt{n}\ (\gamma_n(\tau)-\tau)|;\ 0 \le \tau \le s\}$$

(7.60)

$$+ M_b' \sup\{|V_n(\tau)|;\ 0 \le \tau \le s\}$$

$$\le M_b''(B(s) + B(\Gamma_n^{-1}(s)))q(s)[1/(\sqrt{n}\ q(s))$$

$$+ \sup\{|V_n(\tau)|;\ 0 \le \tau \le s\}/q(s)]$$

where $q(s) = [s(1-s)]^{1/2-\delta/2}$. Obviously M_b''' may be chosen large enough for (7.60) to hold for all n. Now suppose s is in $[b,1)$ and choose δ' such that $0 < b-\delta'$. Then for n exceeding some $n_{b,\delta',\omega}$ we have by Lemma 4 that $W_n^{\#}(s) = \sup\{W_n(\tau);\ b-\delta' \le \tau \le s\}$ and $g_n^{\#}(s) = \sup\{g_n(\tau);\ b-\delta' \le \tau \le s\}$. Thus, for $n > n_{b,\delta',\omega}$ we have

$$|Z_n^{\#}(s)| \le \sqrt{n}\,\sup\{|W_n(\tau)-g_n(\tau)|;\ b-\delta' \le \tau \le s\}$$

$$\le \sup\left\{\left|\frac{\partial\Psi^{-1}}{\partial\xi}(L/(1-\gamma_n(\tau)),\ \xi(\tau))\right|\left|V_n(\tau)\right|;\ b-\delta' \le \tau \le s\right\} \tag{7.61}$$

for some $\xi(\tau)$ between τ and $\Gamma_n^{-1}(\tau)$ where the "sup" is over τ for which the points $(L/(1-\gamma_n(\tau)),\ \xi(\tau))$ lie in $\bar{S}$. Now by the properties of $B(s)$ we observe that

$$B(\xi(\tau)) \le B(\tau) + B(\Gamma_n^{-1}(\tau)). \tag{7.62}$$

Hence, for $n > n_{b,\delta',\omega}$ we have by A5 the second important bound

$$|Z_n^{\#}(s)| \le \sup\{(B(\tau)+B(\Gamma_n^{-1}(\tau)))q(\tau);\ b-\delta' \le \tau \le s\} \cdot \sup\{|V_n(\tau)|/q(\tau);\ b-\delta' \le \tau \le s\} \tag{7.63}$$

for s in $[b,1)$. Now because of the "*" restriction $Z_n^{\#*}(s)$ is zero on $[0,1/n)\cup(1-1/n,1]$. But on $[1/n,b]$ we have

(7.64) $$\sup\{|V_n(\tau)|;\ 0 \le \tau \le s\}/q(s) + 1/(q(s)\sqrt{n})$$

$$\le M_b'''\sup\{|V_n^*(\tau)|/q(\tau);\ 0 \le \tau \le s\} + (nV_1 + 2)/(\sqrt{n}\, q(1/n))$$

$$= O_p(1)$$

by R3 and the fact that nV_1 is $O_p(1)$. Thus, on $(0,b]$ we have for $n > n''_{b,\omega}$ that

(7.65) $$|Z_n^{\#*}(s)| \le (B(s) + B(\Gamma_n^{-1}(s)))q(s)O_p(1)$$

by (7.60) and (7.64) and in fact, this holds for all n. Also, on the sets $S_{n,\varepsilon/2}$ of R4 where $P(S_{n,\varepsilon/2}) < 1-\varepsilon/2$ we have $B(s) + B(\Gamma_n^{-1}(s)) \le 2M_\beta B(s)$. Consequently, for s in $(0,b]$ we obtain

(7.66) $$|Z_n^{\#*}(s)| \le B(s)q(s)O_p(1)$$

on the subset $S_{n,\varepsilon/2}$. Now for s in $[b,1-1/n]$ we have by R3 that $\sup\{|V_n(\tau)|/q(\tau);\ b-\delta' \le \tau \le s\} = O_p(1)$. Also, $B(s)q(s)$ grows unbounded as $s \to 1$ since $b_2 \ge 1$. Thus, on the subset $S_{n,\varepsilon/2}$ **we may replace** $B(\tau) + B(\Gamma_n^{-1}(\tau))$ by $2M_\beta B(\tau)$ in (7.63) and we obtain (7.66) again for $n \ge 1$ and s in $[b,1)$. Thus, we may also construct subsets $\bar{S}_{n,\varepsilon}$ satisfying $\bar{S}_{n,\varepsilon} \subset S_{n,\varepsilon/2}$ and $P(\bar{S}_{n,\varepsilon}) > 1-\varepsilon$ such that (7.53) holds for $\bar{M}_\varepsilon$ chosen sufficiently large.

Theorem 2 (Case I) Fix y in $(0,1]$. Then as $n \to \infty$,

$$T_n(y) \xrightarrow[p]{} T_I(y),$$

with $t_n(y)$ of (3.20) and $\sigma^2(y)$ of (4.14) finite.

<u>Proof</u>: From (3.19) and (3.21) we have for $n > 1/y$

$$T_n(y) = -\int_0^1 Z_n^{\#*}(s \wedge y)\phi'(s)ds - R_{n1} + R_{n2}^y + R_{n3} + R_{n4}^y \tag{7.67}$$

where R_{n1} and R_{n3} are as in Lemma 9, and

$$R_{n2}^y = \begin{cases} \sqrt{n}\,\phi((n-1)/n)\max\{W_i;\ i=1,\dots,n\} & \text{for } \frac{n-1}{n} < y \le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{7.68}$$

and

$$R_{n4}^y = \begin{cases} \sqrt{n}\int_{(n-1)/n}^1 \phi'(s)g_n^{\#}(s\wedge y)ds & \text{for } \frac{n-1}{n} < y \le 1, \\ 0, \text{ otherwise.} \end{cases} \tag{7.69}$$

Also, by A1 we have $0 \le g_n^{\#}(s) \le \Psi^{-1}(L,s)$ on $[0,1)$. Suppose $y \ne y_1^{\#}$. Then by (4.12), (7.67), (7.68) and (7.69) we have

$$|T_n(y)-T_I(y)| \le \int_0^1 |Z_n^{\#*}(s \wedge y)-Z(y^{\#}(s\wedge y))|\,|\phi'(s)|ds + \sum_{i=1}^{4}|R_{ni}| \tag{7.70}$$

since $0 \le R_{n2}^y \le R_{n2}$ and $0 \le |R_{n4}^y| \le |R_{n4}|$ (and this holds in fact for $n \ge 1$). Now by Lemma 5 we have $|Z_n^{\#*}(s \wedge y)-Z(y^{\#}(s\wedge y))| \xrightarrow[e]{} 0$ as $n \to \infty$ for s in $(0,y_1^{\#}) \cup (y_1^{\#},1)$. Also, $R_{ni} \xrightarrow[p]{} 0$ by Lemma 9. To conclude $|T_n(y)-T_I(y)| \xrightarrow[p]{} 0$ we will use the dominated convergence theorem upon showing that $|Z_n^{\#*}(s \wedge y)-Z(y^{\#}(s \wedge y))|\,|\phi'(s)|$ is bounded on $(0,y_1^{\#}) \cup (y_1^{\#},1)$ by an integrable function when ω is a member of a suitably defined set of arbitrarily high probability. By A5 and R3 we have for s in $(0,y_1^{\#}) \cup (y_1^{\#},1)$ that

$$|Z\ (y^{\#}(s))\ | = g^{\partial}(y^{\#}(s))|V(y^{\#}(s))|$$

(7.71) $$\leq B(y^{\#}(s))q(y^{\#}(s))|V(y^{\#}(s))|/q(y^{\#}(s))$$

$$\leq B(s)q(s)O_p(1).$$

Thus, there exists a subset $\bar{S}_{\varepsilon/2}$ such that

(7.72) $$\chi(\bar{S}_{\varepsilon/2})|Z\ (y^{\#}(s))| \leq \bar{M}'_{\varepsilon}B(s)q(s) \quad \text{for} \quad s \quad \text{in} \quad (0,y_1^{\#})\cup(y_1^{\#},1)$$

where $\bar{M}'_{\varepsilon}$ is some positive constant. Hence, let $\bar{S}'_{n,\varepsilon} = \bar{S}_{n,\varepsilon/2} \cap \bar{S}_{\varepsilon/2}$ where $\bar{S}_{n,\varepsilon/2}$ are the sets of Lemma 10, and observe that

$$\chi(\bar{S}'_{n,\varepsilon})|Z_n^{\#*}(s)-Z\ (y^{\#}(s))|$$

(7.73) $$\leq (\bar{M}_{\varepsilon} + \bar{M}'_{\varepsilon})B(s)q(s) \quad \text{for} \quad s \quad \text{in} \quad (0,y_1^{\#})\cup(y_1^{\#},1),$$

and $P(\bar{S}'_{n,\varepsilon}) > 1-\varepsilon$. We may now use the dominated convergence theorem on (7.70). Recall B2 and observe that $\int_0^1 B(s \wedge y)q(s \wedge y)D(s)ds < \infty$ for y in $(0,1]$. Apply the theorem once for each ω to conclude $|T_n(y)-T_I(y)| \underset{p}{\to} 0$. Next suppose $y = y_1^{\#}$. Then by (4.12), (7.67), (7.68) and (7.69) we have for $n > 1/y_1^{\#}$

$$|T_n(y_1^{\#})-T_I(y_1^{\#})| \leq |Z_n^{\#*}(y_1^{\#})-g^{\partial}(y_1^{\#})^{+}\max(V(y_1^{\#}),0)|\phi(y_1^{\#})$$

(7.74) $$+ \int_0^{y_1^{\#}} |Z_n^{\#*}(s)||\phi'(s)|ds + \sum_{i=1}^{4} |R_{ni}|.$$

By Lemmas 6 and 9, the first and last terms of the right hand side converge to zero in probability. In fact we have just demonstrated that the middle term does also since $Z(y^{\#}(s))$ is zero on $(0,y_1^{\#})$.

Finally, the finiteness of $t_n(y)$ given by (3.20) is easy since $g_n^{\#}(s) \leq \Psi^{-1}(L, s)$ for $0 \leq s < 1$ and by A5

$$\Psi^{-1}(L, s) \leq \begin{cases} M_1 + M_2(1-s)^{-b_2+1} & \text{for } b_2 \neq 1 \\ M_1 - M_2\log(1-s) & \text{for } b_2 = 1 \end{cases} \tag{7.75}$$

for some constants M_1 and M_2. Recall $|\phi'(s)| \leq D(s)$ and obtain a finite bounding integral for (3.20). The finiteness of $\sigma^2(y)$ given by (4.14), is obtained upon noting that $\Gamma(s_1,s_2) \leq s_1 g^{\partial}(s_1)g^{\partial}(s_2)(1-s_2)$ for $s_1 < s_2$ in R, and that the integrand in (4.14) is symmetric in s_1 and s_2. Hence, replace $\int_0^1 \int_0^1$ with $2\int_0^1 \int_0^{s_2}$, recall A5 and B2, and obtain a bounding integral which is finite. The proof is complete.

Theorem 3 (Case II) Fix y in $(0,1]$. Then as $n \to \infty$

$$T_n(y) \underset{p}{\to} T_{II}(y) \tag{7.76}$$

and $t_n(y) \to 0$.

Proof. Suppose y is in $(0,y_2^{\#})$. Then by (4.15) and (3.21) we have for $n > 1/y$

$$|T_n(y)-T_{II}(y)| \leq \int_0^1 |Z_n^{\#*}(s \wedge y)||\phi'(s)|ds + \sum_{i=1}^{4} |R_{ni}|. \tag{7.77}$$

By Lemma 7 we have $Z_n^{\#*}(s\wedge y) \underset{e}{\to} 0$ as $n \to \infty$ and by Lemma 9 we have $\sum_{i=1}^{4} |R_{ni}| \underset{p}{\to} 0$. Recalling Lemma 10 we again use the dominated convergence theorem as we did in the proof of Theorem 2 to conclude $|T_n(y)-T_{II}(y)| \underset{p}{\to} 0$. On the other hand suppose y is in $[y_2^{\#}, 1]$. Then we obtain for $n > 1/y$

$$|T_n(y)-T_{II}(y)| \leq \int_0^{y_2^\#} |Z_n^{\#*}(s)||\phi'(s)|ds + \sum_{i=1}^{4} |R_{ni}|$$

$$(7.78) \qquad + \int_{y_2^\#}^{1} |Z_n^{\#*}(s \wedge y)-g^{\partial}(y_2^\#)^+ \max(V(y_2^\#),0)||\phi'(s)|ds.$$

Now by Lemma 7 the integrands converge to zero for fixed s. By Lemma 9 we have $\sum_{i=1}^{4} |R_{ni}| \to 0$ in probability as $n \to \infty$. Thus we need integrable bounding functions for the integrands in (7.78), in order to use the dominated convergence theorem. The approach is basically the same as that in the proof of Theorem 2. The analog of (7.71) holds namely

$$(7.79) \qquad |g^{\partial}(y_2^\#)^+ \max(V(y_2^\#),0)| \leq B(s)q(s)O_p(1).$$

We thus complete the analog of (7.73) which is

$$(7.80) \qquad \chi(\bar{S}'_{n,\varepsilon}) \; |Z_n^{\#*}(s)-g^{\partial}(y_2^\#)^+ \max(V(y_2^\#),0)|$$

$$\leq (\bar{M}_\varepsilon + \bar{M}'_\varepsilon)B(s)q(s) \quad \text{for } s \text{ in } (0,1)$$

and $|T_n(y)-T_{II}(y)| \underset{p}{\to} 0$ as $n \to \infty$ follows. Finally, to establish $t_n(y) \to 0$ as $n \to \infty$, where $t_n(y)$ is given by (3.20), we recall (7.4) and note that $g^\#(y) = 0$ on [0,1). Also $g_n(s) \leq \Psi^{-1}(L,s)$ where $\Psi^{-1}(L,s)$ satisfies (7.75), and $|\phi'(s)| \leq D(s)$. The portion of the integral for $t_n(y)$ on $[0,1-\varepsilon]$ shrinks to zero as $n \to \infty$. The remaining portion on $[1-\varepsilon,1]$ is easily bounded, and this bound is made arbitrarily small by shrinking ε.

<u>Theorem 4</u> (Case III) Fix y in $(0,1]$. As $n \to \infty$

$$(7.81) \qquad T_n(y) \underset{p}{\to} T_{III}(y)$$

and $t_n(y) \to 0$.

Proof: We have for $n > 1/y$

$$|T_n(y)-T_{III}(y)| \leq \int_0^1 |Z_n^{\#*}(s)||\phi'(s\wedge y)||\phi'(s)|ds + \sum_{i=1}^{4} |R_{ni}|. \tag{7.82}$$

By Lemma 9 we have $\sum_{i=1}^{4} |R_{ni}| \xrightarrow[p]{} 0$ as $n \to \infty$ and by Lemma 8 the integrand of (7.82) converges to zero for each s. To conclude convergence in probability we essentially repeat the method of the proof of Theorem 2 in view of Lemma 10. To conclude $t_n(y) \to 0$ as $n \to \infty$, repeat the appropriate portion of the proof of Theorem 4.

In practical applications, one invariably will wish to compute $t_\infty(y)$ of (4.17) rather than $t_n(y)$ of (3.20). Thus we have the following.

Lemma 11 Fix y in (0,1]. As $n \to \infty$,

$$\sqrt{n}\,(t_\infty(y)-t_n(y)) \to 0. \tag{7.83}$$

Proof. We easily establish that

$$|\sqrt{n}\,(t_\infty(y)-t_n(y))| \leq \sqrt{n}\int_{1/n}^{(n-1)/n} |g_n^{\#}(s\wedge y)-g^{\#}(s\wedge y)||\phi'(s)|ds \tag{7.84}$$

$$+ 2|R_{n3}| + 2|R_{n4}|.$$

By the methods of the proof of Lemma 10 we establish

$$(7.85) \qquad \sqrt{n}\, |g_n^{\#}(s)-g^{\#}(s)| \le \sup\{\sqrt{n}\, |g_n(\tau)-g(\tau)|;\ 0 \le \tau \le s\}$$
$$\le MB(s)q(s)\sup\{\sqrt{n}\, |\gamma_n(\tau)-\tau|/q(s);\ 0 \le \tau \le s\}.$$

Now for $1/n \le s \le (n-1)/n$

$$\sup\{\sqrt{n}\, |\gamma_n(\tau)-\tau|/q(s);\ 0 \le \tau \le s\}$$
$$(7.86) \qquad \le \sup\{\sqrt{n}\, |\gamma_n(\tau)-\tau|/q(1/n);\ 0 \le \tau \le s\}$$
$$= 1/(q(1/n)\sqrt{n}) \to 0 \quad \text{as} \quad n \to \infty$$

by (6.6). From Lemma 9 we have $|R_{n3}| \to 0$ and $|R_{n4}| \to 0$ as $n \to \infty$ and we know that $\int_{1/n}^{(n-1)/n} B(s \wedge y)q(s \wedge y)D(s)ds < \infty$ so that (7.83) follows.

Corollary 5: Let $\bar{T}_n(y) = \sqrt{n}\,\{T_n(y)-t_\infty(y)\}$. Then for y in $(0,1]$, Theorems 2 to 4 hold with $\bar{T}_n(y)$ in place of $T_n(y)$.

Proof. Recall $T_n(y) = \sqrt{n}\,(T_n(y)-t_n(y))$ and use Lemma 10.

8. DISCUSSION AND EXAMPLES

We return to the bundle example of section 2 where $\ell_1(t) = L > 0$ and n was taken as infinite. We consider the failure behavior in light of the asymptotic results of the previous section. When the shape function Ψ was Ψ_1 of (1.4) we found Ψ_1^{-1} and g to be given by (2.5) and (2.13) respectively. Examining $\partial\Psi_1^{-1}(x,y)/\partial y$ for $x > 0$, in view of requirement A5, we find $b_1 = 1/2$ and $b_2 = 1+\delta$ suffice in the bounding functions B and D. Thus, for the power law breakdown rule κ_1 of (1.3) we have $\phi(y)$ given by (2.9) and $\rho > 1/2$ is required to satisfy requirement B2 (whereas the finiteness of $t_\infty(1)$ only required $\rho > 0$). In applications, $\rho > 1$ is almost always the case. We also find $\partial\Psi^{-1}(x,y)/\partial x$ easily satisfies A6 and in fact, all the technical assumptions are satisfied when $\Psi = \Psi_1$. Now when $L < L^*$ we have the Case I satuation. The time to failure $T_{[ny+1]}$ of the <u>positive fraction</u> y of fibers in the bundle has the following character. When $0 < y < y_1^\#$ we find $t_\infty(y) = 0$ and $\sigma^2(y) = 0$ so that $\bar{T}_n(y) = \sqrt{n}\, T_{[ny+1]}$ converges to zero in probability. This is consistent with the infinite bundle interpretation. Now for $y = y_1^\# > 0$ we have $t_\infty(y_1^\#) = 0$ again. However $\bar{T}_n(y_1^\#) = \sqrt{n}\, T_{[ny_1^\#+1]}$ is asymptotically normal <u>but with the probability on the negative time axis moved to time zero</u>. Also, $\sigma^2(y_1^\#)$ is positive. Of course this result is consistent with $T_{[ny+1]} \geq 0$ but we may interpret the result further. Upon application of the load L on a large finite bundle the actual fraction to fail may be slightly less than $y_1^\#$ in which case a small amount of time is required for the failed fraction to reach $y_1^\#$. On the other hand if the failed fraction exceeds $y_1^\#$ initially (with probability approaching 1/2 as $n \to \infty$) then $T_{[ny_1^\#+1]}$ is automatically zero. Note that $T_{[ny_1^\#+1]} \to_p 0$ as $n \to \infty$. Now when $y_1^\# < y < y_2^\#$ we find asymptotic normality of $\bar{T}_n(y) = \sqrt{n}\,(T_{[ny+1]} - t_\infty(y))$ and $T_{[ny+1]} \to_p t_\infty(y)$ as $n \to \infty$. But when $y_2^\# \leq y < 1$ we find that $t_\infty(y) = t_\infty(y_2^\#) = t_\infty(1)$ and $\sigma^2(y) = \sigma^2(y_2^\#) = \sigma^2(1)$. Thus $T_{[ny+1]}$

and $T_n(1) = T_n$ are both asymptotically normal with the same mean $t_\infty(1)$ and the same variance $\sigma^2(1)$. This is consistent with the infinite bundle concept that $y_2^{\#}$ was the collapse fraction and the remaining fraction $1 - y_2^{\#}$ fails instantaneously when $y_2^{\#}$ is reached. When $L = L^*$ we have the Case II situation with $t_\infty(y) = 0$ on $(0,1]$. For $0 < y < y_1^{\#} = y_2^{\#}$ we find $\sqrt{n}\, T_{[ny+1]} \to 0$ in probability but when $y_2^{\#} \leq y < 1$, we find that $\sqrt{n}\, T_{[ny+1]}$ is asymptotically normal ($t_\infty(y_2^{\#}) = 0$, $\sigma^2(y_2^{\#}) > 0$) <u>but with the probability on the negative time axis moved to time zero</u>. Also $\sqrt{n}\, T_n(1)$ has the same asymptotic distribution as $\sqrt{n}\, T_{[ny+1]}$ for $y_2^{\#} \leq y < 1$. This is consistent with Daniels' asymptotic result for classic bundle strength. Recall that the initial bundle strength was asymptotically normal with mean L^* and variance which decreased as $1/\sqrt{n}$. Hence upon application of $L = L^*$ the initial strength may be exceeded (with probability approaching 1/2 as $n \to \infty$) in which case the bundle collapses immediately, or its initial strength may exceed L^* in which case a small amount of time will be required for collapse to occur. Finally for $L > L^*$ we obtain $\sqrt{n}\, T_{[ny+1]} \to_p 0$ for $0 < y < 1$ and $\sqrt{n}\, T_n(1) \to_p 0$ as $n \to \infty$.

These general features carry over to many other practical examples of the shape function. The assumptions listed in Section 6 are not restrictive.

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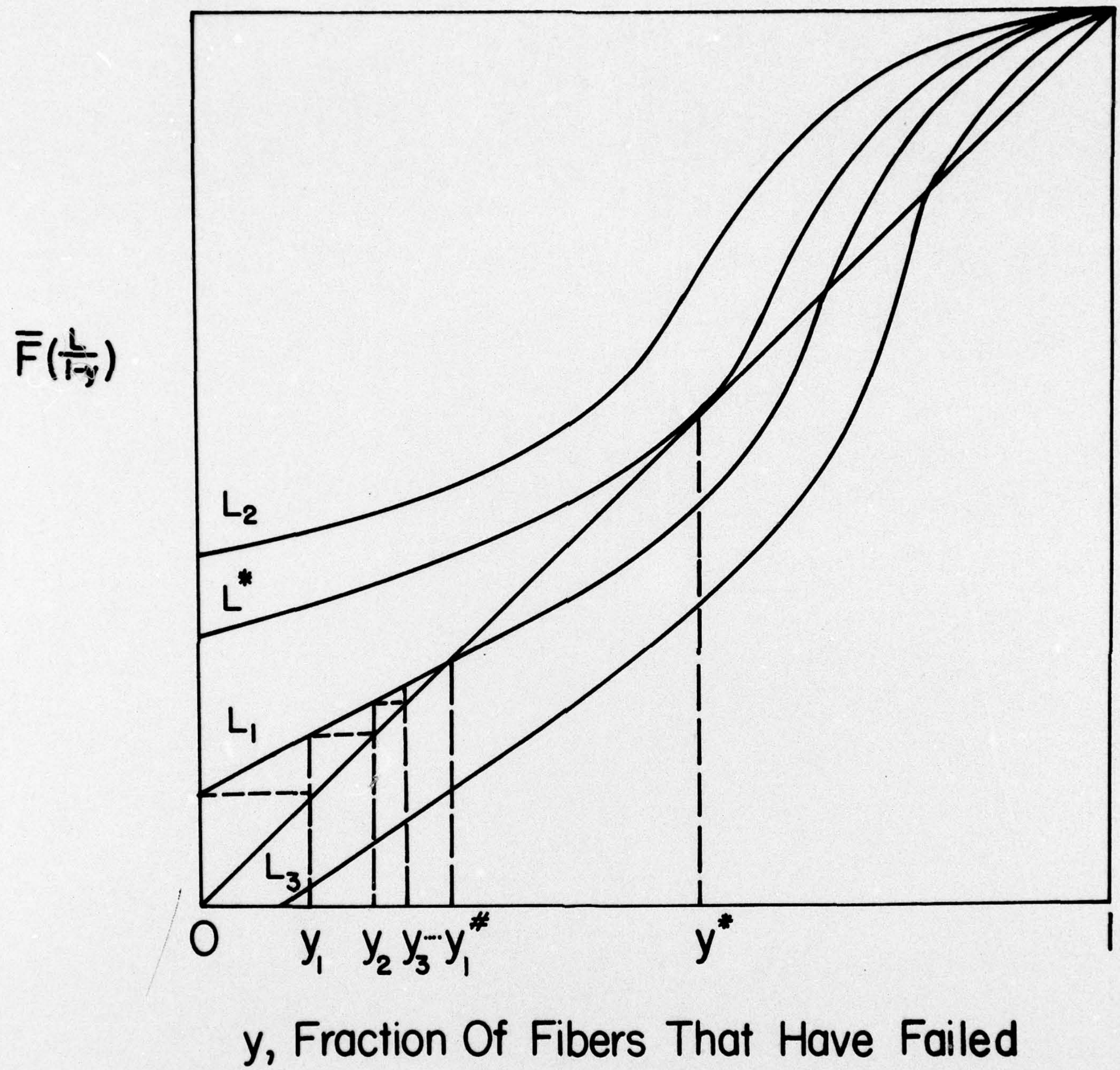

Figure 1. Graph to determine fraction of fibers that fail at time zero.

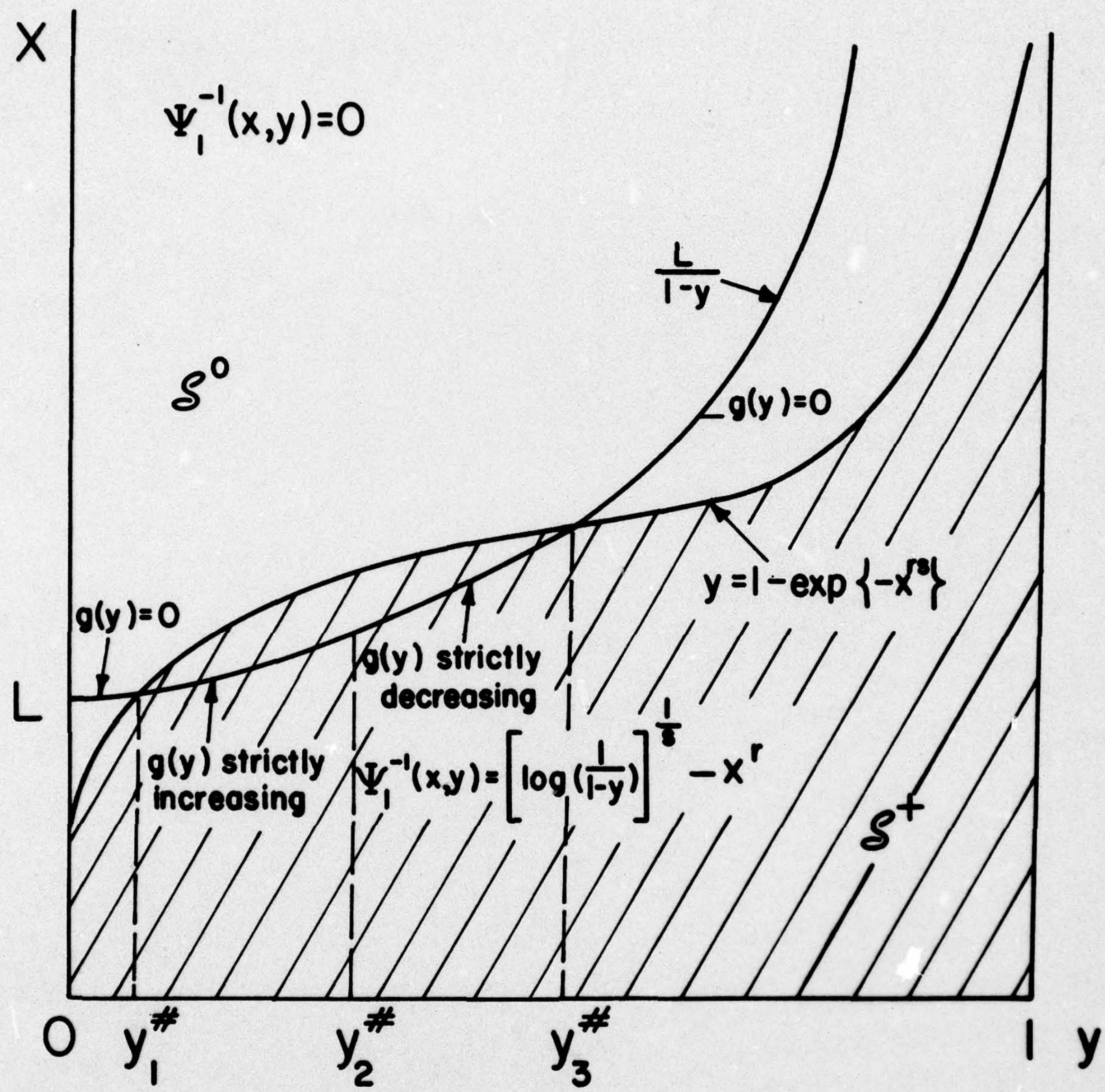

Figure 2. Characteristics of Ψ_1^{-1} and associated functions and sets.